

**UNICAMP**

UNIVERSIDADE ESTADUAL DE  
CAMPINAS

Instituto de Física Gleb Wataghin

HENRIQUE DIAS TRURAN

**Quantum Field Theory in Accelerated  
Reference Frames**

**Teoria Quântica de Campos em Referenciais  
Acelerados**

Campinas

2017

Henrique Dias Truran

# **Quantum Field Theory in Accelerated Reference Frames**

## **Teoria Quântica de Campos em Referenciais Acelerados**

Dissertação apresentada ao Instituto de Física Gleb Wataghin da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Física.

Dissertation presented to the Institute of Physics Gleb Wataghin of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Physics.

Supervisor: Donato Giorgio Torrieri

Este exemplar corresponde à versão final da Dissertação defendida pelo aluno Henrique Dias Truran e orientada pelo Prof. Dr. Donato Giorgio Torrieri.

Campinas

2017

**Agência(s) de fomento e nº(s) de processo(s):** CNPq, 132651/2015-7

Ficha catalográfica  
Universidade Estadual de Campinas  
Biblioteca do Instituto de Física Gleb Wataghin  
Lucimeire de Oliveira Silva da Rocha - CRB 8/9174

T774q Truran, Henrique Dias, 1992-  
Quantum field theory in accelerated reference frames / Henrique Dias  
Truran. – Campinas, SP : [s.n.], 2017.

Orientador: Donato Giorgio Torrieri.  
Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Física Gleb Wataghin.

1. Teoria quântica de campos. I. Torrieri, Donato Giorgio, 1975-. II.  
Universidade Estadual de Campinas. Instituto de Física Gleb Wataghin. III.  
Título.

Informações para Biblioteca Digital

**Título em outro idioma:** Teoria quântica de campos em referenciais acelerados

**Palavras-chave em inglês:**

Quantum field theory

**Área de concentração:** Física

**Titulação:** Mestre em Física

**Banca examinadora:**

Donato Giorgio Torrieri [Orientador]

Pedro Cunha de Holanda

Jorge Jose Leite Noronha Junior

**Data de defesa:** 28-06-2017

**Programa de Pós-Graduação:** Física

MEMBROS DA COMISSÃO JULGADORA DA DISSERTAÇÃO DE MESTRADO DE **HENRIQUE DIAS TRURAN - RA 117204** APRESENTADA E APROVADA AO INSTITUTO DE FÍSICA “GLEB WATAGHIN”, DA UNIVERSIDADE ESTADUAL DE CAMPINAS, EM 28 / 06 / 2017.

**COMISSÃO JULGADORA:**

- **Prof. Dr. Donato Giorgio Torrieri – Orientador – DRCC/IFGW/UNICAMP**
- **Prof. Dr. Pedro Cunha de Holanda – DRCC/IFGW/UNICAMP**
- **Prof. Dr. Jorge José Leite Noronha Junior – IF/USP**

**OBS.:** Informo que as assinaturas dos respectivos professores membros da banca constam na ata de defesa já juntada no processo vida acadêmica do aluno.

CAMPINAS  
2017

# Acknowledgements

Firstly, I would like to express my deepest gratitude to my advisor Prof. Giorgio Torrieri for his incredible support during my Master's study and research. His patience, good humor and motivation consistently helped me to keep moving forward and keep learning new things. His guidance and mentoring taught me what it means to do research, and for that I am forever grateful.

I would also like to thank all the people who kindly participated in all the committee related to this work during my course: Prof. Jun Takahashi, Prof. Alberto Saa, Prof. Arlene Aguilar, Prof. Marcelo Guzzo, Prof. Jorge Noronha and Prof. Pedro Holanda. Their insightful comments and thought-provoking questions were of invaluable help in this thesis and at illuminating the path to future research. A special thanks to Lance Labun, whose conversations and discussions with me and Prof. Giorgio were essential to the development of this thesis.

I would also like to thank all my family and friends for their incredible support and continuous encouragement during all these years of study. I will be forever indebted to all the entertaining Friday night conversations with my parents, to all the times I went to movie theater with my brother to watch whatever was on screen, and to the occasional night-out madness with my friends.

Lastly, I would like thank CNPq (bolsa de mestrado 132651/2015-7) for their financial support during the progress of this work.

# Resumo

Teoria quântica de campos é uma das mais bem sucedidas teorias da física contemporânea, e suas aplicações no estudo de sistemas acelerados um interessante tópico de pesquisa. De termalização em física de íons pesados à radiação de elétrons em QED de alta intensidade, o estudo de TQC em referenciais acelerados parece mais relevante do que nunca. Nessa dissertação nós apresentamos uma introdução aos princípios da teoria quântica de campos, como podemos estudá-la em referenciais acelerados, e como relacionar a física que um observador acelerado vê com os fenômenos observados no referencial inercial do laboratório.

**Palavras-chave:** teoria quântica de campos.

# Abstract

Quantum field theory is one of the most successful theories of contemporary physics, and its applications to the study of accelerated systems an interesting topic of research. From thermalization in heavy-ion physics to radiation emission by electrons in high-intensity QED, the study of QFT in accelerated reference frames seems more relevant than ever. In this thesis we present an introduction to the principles quantum field theory, how we can study it in accelerated reference frames, and how to relate the physics seen by an accelerated observer's point of view with the phenomena observed in the inertial frame of the laboratory.

**Keywords:** quantum field theory.

# Contents

<b>1</b>	<b>MOTIVATION</b>	<b>10</b>
<b>2</b>	<b>QUANTUM FIELD THEORY</b>	<b>13</b>
<b>2.1</b>	<b>Classical Fields</b>	<b>13</b>
2.1.1	General Structure	13
2.1.2	Relativistic Fields	14
2.1.2.1	Minkowski Space	14
2.1.3	Example: Scalar Field	15
<b>2.2</b>	<b>Canonical Quantization</b>	<b>17</b>
2.2.1	Scalar Field Quantization	18
2.2.2	Physical Interpretation and Observables	20
<b>2.3</b>	<b>Path Integral Quantization</b>	<b>22</b>
2.3.1	Path Integrals in Quantum Mechanics	22
2.3.2	Path Integrals in QFT	24
2.3.3	Scalar Field Quantization	25
<b>2.4</b>	<b>Finite Temperature QFT</b>	<b>26</b>
2.4.1	Fields and Temperature	27
2.4.2	Thermo Field Dynamics	28
2.4.3	Path-Integral Formulation of TFD	30
<b>2.5</b>	<b>QFT in General Coordinate Systems</b>	<b>31</b>
2.5.1	Scalar Field on Curved Spacetime	32
<b>3</b>	<b>THE UNRUH EFFECT</b>	<b>34</b>
<b>3.1</b>	<b>Preliminaries</b>	<b>34</b>
<b>3.2</b>	<b>Accelerated Observer</b>	<b>35</b>
<b>3.3</b>	<b>Free Fields in Rindler Space</b>	<b>36</b>
3.3.1	Scalar Field	37
3.3.2	Bogoliubov Transformation	39
3.3.3	Rindler Particles in Minkowski Vacuum	41
<b>3.4</b>	<b>Interacting Fields in Rindler Space</b>	<b>42</b>
<b>4</b>	<b>RINDLER HORIZON AND THERMODYNAMICS</b>	<b>45</b>
<b>4.1</b>	<b>Rindler Horizon and Thermo Field Dynamics</b>	<b>46</b>
4.1.1	Scalar Field Propagator in Rindler Coordinates	46
4.1.2	Scalar Field Propagator in Thermo Field Dynamics	49
<b>4.2</b>	<b>Example: Internal Particle Detectors</b>	<b>53</b>



4.2.1	Detector Model . . . . .	53
4.2.2	Thermo Field Dynamical Approach . . . . .	54
5	<b>CONCLUSIONS AND DISCUSSIONS . . . . .</b>	<b>57</b>
	<b>BIBLIOGRAPHY . . . . .</b>	<b>59</b>

# Chapter 1

## Motivation

The Quantum theory of fields is perhaps one the most important tools in modern physics, being the underlying framework of our current understanding of elementary particles and an invaluable support to other areas such as nuclear, atomic and condensed matter physics. Experimental measurements of quantum electrodynamical observables, such as the fine structure constant  $\alpha$ , makes QED the most precise theory in the history of science. Despite all of its success and importance, there are still a lot of areas that remain, to this day, far from understood.

One of such topics is the phenomenology of thermalization in hadronic collisions [1]. Experimentally, it is observed that in high-energy heavy-ion collisions large systems seem to thermalize very fast [2, 3]. Although not surprising, since large systems are expected to be well described by statistical mechanics [4], the time scale of such thermalization mechanism cannot be explained from first principles. A greater mystery perhaps is that even small systems, such as proton-nucleous, proton-proton and even electron-positron seem to show the same thermal behavior in high-energy collisions [5]. More than that: they seem to thermalize at the same temperature  $T \sim 170$  MeV, independent of their center of mass energy.

This apparently “universal” thermal behavior of heavy-ion collisions presents a challenge for the theory, and is an extremely active field of research. Recently, there has been attempts to explain this phenomena based on a theoretical prediction of quantum field theory - the Unruh effect [6, 7, 8, 9]. First described by Fulling, Davies and Unruh in the 70’s [10, 11], the Unruh effect is the statement that an accelerated observer sees the vacuum as a thermal bath of particles whose temperature is proportional to its acceleration. These proposals attempt to explain the observed thermal hadronization of small systems by the Unruh-temperature they experience in their rest frame.

Such thermalization mechanisms, however, rely on the contribution of the Unruh thermal bath to the dynamical evolution of the system up to hadronization. This raises the question: how to explain such thermalization mechanism from the point-of-view of the

laboratory, which for all intents and purposes, is a perfectly inertial reference frame? This relationship between physics from the point of view of inertial and accelerated observers was studied for a variety of semi-classical systems [12, 13, 14]. In all of these analyses, the study of the systems from an accelerated observer's point of view was shown to be entirely equivalent to the study of the system in inertial frames when taken into account the Unruh thermal bath.

To understand how these Unruh-thermalization mechanisms translate to the laboratory frame is then equivalent to understanding how we can relate inertial and accelerated points of view. More than that, this relationship is essential for a solid understanding of semi-classical accelerated systems in general. Many other ideas, such as radiation emitted by accelerated electrons for example [15], ground their intuition on the way the accelerated particle sees the world from their point of view. For such proposals, the relationship between inertial and accelerated processes is the translation of our physical intuition in the particle's rest frame to the experimental observations performed in the laboratory.

In this thesis, we shall attempt study this relationship in a somewhat more systematic way. In order to do this, we attempt to relate the two points of view by the perturbative diagrams they write in order to describe physical processes and calculate observables. This approach is an alternative application of Horibe, Hosoya and Yammamoto's perturbative analysis of the Unruh effect [16]. By the end we shall see how the inertial and accelerated descriptions of a system are two sides of the same coin, but acceleration is capable of changing the way one describes a system due to the appearance of a spacetime horizon. This change in description due to horizons is very well known, studied in the context of both accelerated reference frames [17, 18] and quantum field theory in curved spacetime [19, 20, 21]. Given the relationship between perturbative diagrams associated with both observers, we can study accelerated systems from the point of view of their rest frame while maintaining the connection to what kinds of processes the inertial observer sees.

It is important to note that here we are not tackling the problems of thermalization of heavy-ion collisions or radiation by accelerated charges, but merely trying to understand how the analysis of such problems based on the Unruh-effect-intuition of how physics works in accelerated reference frames are connected to the laboratory's description of the physical system in question. It is clear that both approaches should yield the same predictions [22], but intuition and practicality of calculations naturally chooses one over the other.

The thesis is structured as follows: In chapter 2 we describe the basic foundation of quantum field theory, highlighting the dependence of our definition of particle excitations with the Poincaré symmetries of spacetime. We also briefly discuss the formalism of thermo-field dynamics for quantum fields at finite temperature, as well as quantum fields on generalized coordinates, and lay the foundations to our discussion about the Unruh

effect later. In chapter 4 we finally discuss the Unruh effect, and show how the foundations of QFT naturally gives us the thermal nature of the inertial vacuum from the point of view of accelerated particle excitations. Finally in chapter 5 we discuss how to connect the inertial and accelerated descriptions of a system by studying the Feynman propagators on inertial and accelerated reference frames, and present an example of this procedure for a particular field-detector system. We conclude with a brief discussion of the topics covered and future prospects.

## Chapter 2

# Quantum Field Theory

Just as classical field theory is the extension of classical mechanics to continuous systems, quantum field theory is the extension of quantum mechanics to systems with infinitely many degrees of freedom. While classical mechanics is interested in the dynamics of a discrete number of coordinates  $q_i$ , classical field theory is interested in the dynamics of fields  $\phi$  propagating in a background, classical spacetime  $(\mathcal{M}, g)$ , where  $\mathcal{M}$  is a smooth manifold and  $g$  a Lorentzian metric on the tangent spaces  $T_x\mathcal{M}$ . In this formalism, each point  $x \in \mathcal{M}$  in spacetime is now associated with some dynamical field variable which is labeled  $\phi(x)$ . Quantum field theory is thus the general framework that extends quantum mechanics to such systems of fields. Due to its natural capability of reconciling quantum mechanics and special relativity, we shall present the topic in the context of relativistic fields and particle physics.

In this chapter we give a brief overview of the two main ways of extending the quantum mechanical quantization procedure to systems of relativistic fields: The canonical quantization and the path integral procedures. Both quantization programs will be presented in the context of scalar fields for the sake of clarity. We end the chapter with a brief discussion about QFT at finite temperature and in curved-spacetimes and general coordinate systems, which shall be useful when we treat accelerated reference frames on later chapters. We shall adopt throughout the thesis the natural system of units  $c = \hbar = k_B = 1$ .

## 2.1 Classical Fields

### 2.1.1 General Structure

As said previously, classical field theory is interested in the dynamics of a continuous field variable  $\phi(x)$  defined at every point  $x \in \mathcal{M}$  of a given background spacetime  $(\mathcal{M}, g)$ . Such theories consists of four elements:

1. An  $n$ -dimensional Riemannian (or pseudo-Riemannian) manifold  $(\mathcal{M}, g)$  representing

the background, classical spacetime in which the theory is defined.

2. A map  $\phi : \mathcal{M} \rightarrow X$  from the background spacetime  $\mathcal{M}$  to a configuration space  $X$  (usually  $\mathbb{R}^k$  or  $\mathbb{C}^k$ ), called a field, representing the physical system of interest.
3. A functional  $S[\phi]$ , called the action, that determines the dynamics of the field  $\phi$  according to an extremum principle  $\delta S = 0$ . The action is usually written in terms of a Lagrangian density  $\mathcal{L}$ , which is a functional of the field  $\phi$  and its derivatives  $D_\mu \phi$ :

$$S[\phi] = \int d^n x \mathcal{L}[\phi, D_\mu \phi](x) \quad (2.1)$$

4. A family of functionals  $\{O_\alpha : \phi \mapsto \mathbb{R}\}_{\alpha \in \mathcal{I}}$  indexed by an arbitrary set  $\mathcal{I}$ , defined on the set of fields, that represent the observables of the physical system in question.

Given the action functional for a field  $\phi$ , the extremum condition for the action  $\delta S = 0$  gives us a dynamical equation for the field:

$$D_\mu \left( \frac{\delta \mathcal{L}}{\delta (D_\mu \phi)} \right) - \delta_\phi \mathcal{L} = 0 \quad (2.2)$$

which specifies the dynamics of the field given an initial condition  $\phi_0$ .

### 2.1.2 Relativistic Fields

Now we shall characterize the general structure of the previous section to the particular case of relativistic field theories. We start by specifying the background spacetime of special relativity, the Minkowski space, and its group of isometries that characterizes inertial reference frames, the Poincaré group. We proceed to characterize relativistic field theories by the transformation properties of its action functional under the (induced) action of the Poincaré group of transformations in spacetime. Finally we give an example of a relativistic field theory that we are going to use as a toy model throughout the thesis.

#### 2.1.2.1 Minkowski Space

The background, classical spacetime of special relativity, called Minkowski space  $\mathcal{M}_{ink}^4$ , is the (1+3) dimensional flat Lorentzian manifold  $(\mathbb{R}^4, \eta)$  whose spacetime line element is given by

$$d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\mathbf{x}^2 \quad (2.3)$$

The group of spacetime transformations on  $\mathcal{M}_{ink}^4$  that leave the line element  $d\tau^2$  invariant is called the Poincaré Group  $\mathcal{P}$ . This is the group of isometries of Minkowski space, and consists of spacetime translations together with (proper and improper) Lorentz transformations:

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad (2.4)$$

where  $a^\mu \in \mathbb{R}^4$  is a spacetime translation and the linear transformation  $\Lambda : \mathcal{M}_{ink}^4 \rightarrow \mathcal{M}_{ink}^4$  is defined by:

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad (2.5)$$

Einstein's principle of relativity states the equivalence between certain frames of reference, called inertial reference frames. Any frame of reference that can be obtained from an inertial one by means of a Poincaré transformation is itself inertial<sup>1</sup>.

From the principle of relativity, all laws of physics must be the same in all inertial reference frames. Since the action of a (proper, orthochronous)<sup>2</sup> Poincaré transformation  $(\Lambda, a)$  on Minkowski space induces a transformation  $S \rightarrow S'$  on the action functional of a field  $\phi$  defined on  $\mathcal{M}_{ink}^4$ , the equivalence between inertial reference frames implies that for a field theory to be relativistic its action functional  $S[\phi]$  must be invariant under the induced action of the proper, orthochronous Poincaré group  $\mathcal{P}_+^\uparrow$ .

This invariance requirement of the action for relativistic theories constrains the transformation properties of the Lagrangian density  $\mathcal{L}[\phi, D_\mu \phi](x)$  of the theory. A theory is therefore relativistic if its Lagrangian density  $\mathcal{L}$  transforms as a scalar under  $\mathcal{P}_+^\uparrow$ :

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x') \quad (2.6)$$

This characterization allows us to construct relativistic theories by finding Lagrangian densities which transforms as scalars under the action of the Poincaré group<sup>3</sup>. This is done by constructing the theory with fields that, under the induced action of spacetime Poincaré transformations  $(\Lambda, a)$ , transforms as representations of the Poincaré group:

$$\phi(x) \rightarrow U(\Lambda, a)\phi(x') \quad (2.7)$$

Finding all possible relativistic field theories is now reduced to the mathematical problem of finding all representations of the group  $\mathcal{P}_+^\uparrow$ . From the classification of its representations of the concept of mass and spin naturally arises, and they determine the transformation properties of the respective field<sup>4</sup>.

### 2.1.3 Example: Scalar Field

We now present an example of a relativistic field theory based on the simplest representation of the Poincaré group, the (massive) scalar field. We shall present the

<sup>1</sup> This means that the set of all inertial reference frames is an equivalence class under the action of the Poincaré group.

<sup>2</sup> This is the subgroup of the Lorentz transformations  $\Lambda$  satisfying  $\det \Lambda = +1$  and  $\Lambda^0{}_0 \geq 1$ . For relativistic invariance we shall require only invariance of the action under proper orthochronous Lorentz transformations, since QFT need not be invariant under parity nor time reversal. See for example chapters 2 and 3 of [23] and references therein for a discussion about discrete symmetries in QFT and parity violation in weak interactions.

<sup>3</sup> From now on, whenever we talk about the Poincaré group we mean the proper, orthochronous subgroup  $\mathcal{P}_+^\uparrow$ .

<sup>4</sup> For more details on the construction of the representations of  $\mathcal{P}_+^\uparrow$  see [23] chapter 2.

theory, for later convenience, in a general Minkowski dimension  $\mathcal{M}_{ink}^{(1+n)}$ , with metric tensor  $\eta_{\mu\nu}$  of mostly minus signature  $(1, -n)$ .

The theory is constructed from the field  $\phi : \mathcal{M}_{ink}^{(1+n)} \rightarrow \mathbb{R}$ , which transforms under the scalar representation of the Poincaré group:

$$\phi(x) \rightarrow \phi(x') = \phi((\Lambda, a)^{-1}x) \quad (2.8)$$

The scalar Lagrangian density for the field is given by

$$\mathcal{L}[\phi, \partial_\mu \phi](x) = \frac{1}{2} \left( \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi(x)^2 \right) \quad (2.9)$$

The extremum condition on the action functional associated with the Lagrangian density (2.9) establishes the field equation (2.2) for the dynamics of  $\phi$ :

$$\left( \partial^\mu \partial_\mu + m^2 \right) \phi(x) = 0 \quad (2.10)$$

which is called the Klein-Gordon equation. Given initial conditions on the field (and its derivatives) at an initial time  $t_0 = 0$ , one can solve the Klein-Gordon equation and find the field configuration at later times  $t$ .

Sometimes, however, it is more convenient to express the theory in the alternative but equivalent Hamiltonian formulation of classical field theory. This is done by performing a Legendre transformation in the Lagrangian density  $\mathcal{L}$  with respect to the canonical momentum

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi)} \quad (2.11)$$

and obtaining the Hamiltonian density  $\mathcal{H}[\phi, \pi, \nabla \phi](x)$  for the theory:

$$\mathcal{H}[\phi, \pi, \nabla \phi](x) = \pi(x) \partial_t \phi(x) - \mathcal{L}[\phi, \pi, \nabla \phi](x) \quad (2.12)$$

which for the scalar field  $\phi$  is

$$\mathcal{H}[\phi, \pi, \nabla \phi](x) = \frac{1}{2} \left( \pi(x)^2 + (\nabla \phi(x))^2 + m^2 \phi(x)^2 \right) \quad (2.13)$$

The Hamiltonian  $H[\phi, \pi, \nabla \phi](t)$  for the scalar field is given by the integration of  $\mathcal{H}$  over all space:

$$H[\phi, \pi, \nabla \phi](t) = \int d^n \mathbf{x} \mathcal{H}[\phi, \pi, \nabla \phi](t, \mathbf{x}) \quad (2.14)$$

Here it is worth noting that even though the theory constructed is entirely relativistic, manifest Poincaré covariance is lost in the Hamiltonian formalism due to its distinct treatment of the timelike coordinate  $t$ . Both the Lagrangian and Hamiltonian formalism of the theory can be used to extend quantum mechanics to field systems, each with its advantages and disadvantages. The canonical quantization procedure is based on the Hamiltonian formalism of field theory, where manifest Poincaré covariance is the price you pay for a more direct connection with the usual quantization procedure of quantum mechanics. The path integral quantization scheme is based on the Lagrangian formalism, in which manifest unitarity of the theory is the price you pay for the manifest Poincaré covariance of the formalism.



## 2.2 Canonical Quantization

As we discussed in the previous section, the canonical quantization procedure to extend quantum mechanics to systems of fields is based on the Hamiltonian formalism of classical field theory. In usual quantum mechanics, we promote a classical theory to a quantum one by promoting their degrees of freedom to operators acting on a given Hilbert space  $\mathcal{H}$  and imposing suitable commutation relations between them. For a system consisting of a discrete number of particles with degrees of freedom  $x_i$  and  $p_j$ , the commutation relations imposed on the promoted operators  $\hat{x}_i$  and  $\hat{p}_j$  are given by

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \mathbb{I}_{\mathcal{H}} \quad (2.15)$$

where  $\mathbb{I}_{\mathcal{H}}$  is the identity operator of  $\mathcal{H}$ .

We can thus generalize the canonical quantization procedure of quantum mechanics to systems of fields by promoting the fields to operator valued distributions  $\hat{\phi}(t, \mathbf{x})$  and  $\hat{\pi}(t, \mathbf{x})$  acting on a given Hilbert space of states  $\mathcal{H}$ , and imposing canonical commutation relations analogous to (2.15):

$$[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})]_{\pm} = i\delta(\mathbf{x} - \mathbf{y}) \mathbb{I}_{\mathcal{H}} \quad (2.16)$$

where the  $\pm$  index denotes either a commutator for integer spin fields or anti-commutator for half-integer spins.

For free theories<sup>5</sup> one can readily construct representations of the canonical commutation relations (2.16) of the fields as operators acting on an infinite dimensional Fock space of states

$$\mathcal{F}^{\pm} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{\pm} \quad (2.17)$$

where  $\mathcal{H}_n^+(\mathcal{H}_n^-)$  is the symmetric (anti-symmetric) product of  $n$  copies of the single-particle Hilbert space  $\mathcal{H}$

$$\mathcal{H}_n^{\pm} = S^{\pm} \mathcal{H}^{\otimes n} \quad (2.18)$$

and the symmetry properties of the space of states  $\mathcal{F}^{\pm}$  is determined by the spin of the field.

For the general case of interacting theories the construction of such representations of the canonical commutation relations is not yet available<sup>6</sup>. In particular cases of low-dimensional systems<sup>7</sup> such construction is available, but for more general systems one must resort to perturbative [25] or lattice [26, 27] constructions in order to establish foundations for the theory.

<sup>5</sup> That means theories whose Lagrangian densities are at most quadratic in the fields and its derivatives.

<sup>6</sup> In fact, the construction of a mathematically rigorous formulation of quantum Yang-Mills theory – the cornerstone of our current understanding of particle physics – is one of the millennium prize problems presented by the Clay Institute of Mathematics.

<sup>7</sup> Examples of such systems are the massless Thirring model and the Schwinger model of massless QED in (1+1) spacetime dimensions. For more details see [24].

### 2.2.1 Scalar Field Quantization

We shall now proceed to canonically quantize the scalar field theory in  $(1+n)$  dimensions presented in section 2.1.3. As discussed previously, we must promote the fields  $\phi$  and  $\pi$  to operators  $\hat{\phi}$  and  $\hat{\pi}$  defined on a Hilbert space  $\mathcal{H}$  in such a way to satisfy the equal-times commutation relation (2.16).

Before we do this, we first note that the classical solution of the Klein-Gordon equation for the (real) scalar field  $\phi(x)$  can be expanded as

$$\phi(x) = \int d^n \mathbf{k} (A(\mathbf{k}) f_{\mathbf{k}}(x) + A^*(\mathbf{k}) f_{\mathbf{k}}^*(x)) \quad (2.19)$$

where  $A(\mathbf{k}) \in \mathbb{C}$  for  $\mathbf{k} \in \mathbb{R}^n$  are the expansion coefficients and  $f_{\mathbf{k}}(x)$  is the orthonormal solution of the Klein-Gordon equation (2.10)

$$f_{\mathbf{k}}(t, \mathbf{x}) = \frac{e^{i\mathbf{k}\mathbf{x} - iE_{\mathbf{k}}t}}{\sqrt{2E_{\mathbf{k}}(2\pi)^n}} = \frac{e^{-ik_{\mu}x^{\mu}}}{\sqrt{2E_{\mathbf{k}}(2\pi)^n}} \quad (2.20)$$

$$k^{\mu} = (E_{\mathbf{k}}, \mathbf{k}) \quad \text{and} \quad E_{\mathbf{k}} = +\sqrt{\mathbf{k}^2 + m^2} \quad (2.21)$$

with respect to the Klein-Gordon scalar product  $\langle \cdot, \cdot \rangle_{KG}$  defined on the hypersurface of simultaneity  $\Sigma_{t_0} = \{(t, \mathbf{x}) \in \mathcal{M}_{ink}^{(1+n)} \mid t = t_0, t_0 \in \mathbb{R}\}$ :

$$\langle \phi_1, \phi_2 \rangle_{KG} = -i \int_{\Sigma_{t_0}} d^n \mathbf{x} (\phi_1(x) \partial_t \phi_2^*(x) - \phi_2^*(x) \partial_t \phi_1(x)) \quad (2.22)$$

The Klein-Gordon modes  $f_{\mathbf{k}}$  satisfy the relation

$$\partial_t f_{\mathbf{k}}(x) = -iE_{\mathbf{k}} f_{\mathbf{k}}(x) \quad (2.23)$$

with respect to the generator of  $t$ -translations in spacetime  $\partial_t$ . We call the solutions to the Klein-Gordon equation that satisfy this relation “positive frequency” modes with respect to  $t$ .

The solutions  $f_{\mathbf{k}}^*$  satisfy a complementary relation

$$\partial_t f_{\mathbf{k}}^*(x) = iE_{\mathbf{k}} f_{\mathbf{k}}^*(x), \quad (2.24)$$

and they are called “negative frequency” modes with respect to  $t$ .

**Remark.** It is important to note that the definition of positive and negative frequencies is Poincaré invariant: Let  $(\Lambda, a) \in \mathcal{P}_+^{\uparrow}$  be a Poincaré transformation. Since  $k_{\mu}x^{\mu}$  transforms as a scalar, the action of  $(\Lambda, a)$  on  $(t, x) \in \mathcal{M}_{ink}^{(1+n)}$  is simply

$$k_{\mu}x^{\mu} \rightarrow k'_{\mu}x'^{\mu} \quad (2.25)$$

Under such transformation, the modes  $f_{\mathbf{k}}(x')$  and  $f_{\mathbf{k}}^*(x')$  satisfy

$$\partial_{t'} f_{\mathbf{k}}(x') = -iE_{\mathbf{k}} f_{\mathbf{k}}(x') \quad \text{and} \quad \partial_{t'} f_{\mathbf{k}}^*(x') = iE_{\mathbf{k}} f_{\mathbf{k}}^*(x'), \quad (2.26)$$

and thus are positive and negative frequency modes with respect to  $t'$ . The characteristic frequency of the mode is therefore the same across all inertial reference frames. This property is important because later, when we talk about canonical quantization of fields, it is going to ensure the consistency of the vacuum definition across inertial observers.

From the general classical solution (2.19) we can see that promoting the fields  $\phi$  and  $\pi$  to operators is equivalent to promoting the expansion coefficients  $A(\mathbf{k})$  and  $A^*(\mathbf{k})$  to operators

$$A(\mathbf{k}) \rightarrow \hat{a}(\mathbf{k}) \quad \text{and} \quad A^*(\mathbf{k}) \rightarrow \hat{a}^\dagger(\mathbf{k}) \quad (2.27)$$

acting on a Hilbert space of states  $\mathcal{H}$  for the theory. To find a specific representation of the canonical commutation relations (2.16) for the fields in a particular Hilbert space  $\mathcal{H}$  is then equivalent to finding a representation for the new operators  $\hat{a}(\mathbf{k})$  and  $\hat{a}^\dagger(\mathbf{k})$  on the same space of states  $\mathcal{H}$  satisfying:

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}') \quad (2.28)$$

From our experience with the simple harmonic oscillator in non-relativistic quantum mechanics, we can readily construct a representation of the commutation relations for the operators  $\hat{a}(\mathbf{k})$ ,  $\hat{a}^\dagger(\mathbf{k})$ <sup>8</sup>: They act as creation ( $\hat{a}^\dagger$ ) and annihilation ( $\hat{a}$ ) operators on the symmetric Fock space (2.17) of states. Given a state vector  $\Psi_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N} \in \mathcal{F}^+$ , the action of  $\hat{a}^\dagger(\mathbf{k})$  and  $\hat{a}(\mathbf{k})$  can be defined as

$$\hat{a}^\dagger(\mathbf{k}) \Psi_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N} = \Psi_{\mathbf{k} \mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N} \quad (2.29)$$

and<sup>9</sup>

$$\hat{a}(\mathbf{k}) \Psi_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N} = \sum_{i=1}^N \delta(\mathbf{k} - \mathbf{k}_i) \Psi_{\mathbf{k}_1 \dots \mathbf{k}_{i-1} \mathbf{k}_{i+1} \dots \mathbf{k}_N} \quad (2.30)$$

where we can verify that indeed the operator relation (2.28) indeed holds true. Thus the creation and annihilation operators  $\hat{a}^\dagger$  and  $\hat{a}$  induce a representation of the canonical commutation relations (2.16) for the fields  $\hat{\phi}$  and  $\hat{\pi}$  in the Fock space  $\mathcal{F}^+$  of states. The full quantized scalar field can then be expressed as the  $\mathcal{F}^+$ -operator valued distribution

$$\hat{\phi}(x) = \int d^n \mathbf{k} \left( \hat{a}(\mathbf{k}) f_{\mathbf{k}}(x) + \hat{a}^\dagger(\mathbf{k}) f_{\mathbf{k}}^*(x) \right) \quad (2.31)$$

defined on  $\mathcal{M}_{ink}^{(1+n)}$ . The canonical momentum  $\hat{\pi}(x) = \partial_t \hat{\phi}(x)$  can be written as

$$\hat{\pi}(x) = -i \int d^n \mathbf{k} E_{\mathbf{k}} \left( \hat{a}(\mathbf{k}) f_{\mathbf{k}}(x) - \hat{a}^\dagger(\mathbf{k}) f_{\mathbf{k}}^*(x) \right) \quad (2.32)$$

by using the positive and negative frequency properties of  $f_{\mathbf{k}}$  and  $f_{\mathbf{k}}^*$ .

<sup>8</sup> For more details of this construction we refer the reader to chapter 2 of [28] and chapter 4 of [23].

<sup>9</sup> It is worth remembering that in quantum field theory all operators are defined in the sense of distributions, and such expressions are always meant to be smeared over compactly supported functions.

The space of states  $\mathcal{F}^+$  can be constructed entirely from the action of the creation operators in the vacuum state  $\Psi_0$ :

$$\prod_{i=1}^N \hat{a}^\dagger(\mathbf{k}_i) \Psi_0 = \Psi_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N} \quad (2.33)$$

where the vacuum  $\Psi_0 \in \mathcal{F}^+$  is defined to be the unique state which is annihilated by all  $\hat{a}(\mathbf{k})$ :

$$a(\mathbf{k}) \Psi_0 = 0 \quad (2.34)$$

It is usual to employ, just as in standard quantum mechanics, the shorthand Dirac notation  $|\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N\rangle$  for the state vectors  $\Psi_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_N}$  in  $\mathcal{F}^+$ . The vacuum state is written as  $|0\rangle$ , and the inner product in  $\mathcal{F}^+$  is given by

$$\langle \mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_M | \mathbf{q}_1 \mathbf{q}_2 \dots \mathbf{q}_N \rangle = M \delta_{MN} \prod_{i=1}^M \delta(\mathbf{k}_i - \mathbf{q}_i) \quad (2.35)$$

## 2.2.2 Physical Interpretation and Observables

In order to give a physical interpretation for the elements of the Fock space of states  $\mathcal{F}^+$ , we shall study the eigenvalues of the Hamiltonian operator  $H$  defined in (2.14). In terms of the creation and annihilation operators  $\hat{a}^\dagger$  and  $\hat{a}$  the Hamiltonian  $H$  takes the form:

$$H = \int d^n \mathbf{k} \left( E_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) + \frac{1}{2} \delta(0) \right) \quad (2.36)$$

which is diagonal with respect to the Fock states  $|\mathbf{k}_1 \dots \mathbf{k}_N\rangle$ . The eigenvalues of  $H$  are given by a finite part

$$E_{\mathbf{k}} = +\sqrt{k^2 + m^2} \quad (2.37)$$

and an infinite contribution

$$E_0 = \int d^n \mathbf{k} \frac{1}{2} \delta(0) \quad (2.38)$$

due to the zero-point energy of the field being integrated over all space.

Rescaling this divergent contribution  $H \rightarrow H - E_0$  in such a way as to define the energy of the vacuum state to be zero<sup>10</sup>, we now find that the spectrum of the Hamiltonian is precisely the energy of a system of free relativistic particles  $\sum_{i=1}^N E_{\mathbf{k}_i}$ , whose eigenstates are the Fock states  $|\mathbf{k}_1 \dots \mathbf{k}_n\rangle$ .

<sup>10</sup> It is important to note that even though the zero point energy in the absence of gravity has no physical significance, changes in such terms due to changes in the boundary conditions for the fields are physically significant and have even been experimentally tested. We refer the reader to chapter 5 of [29] for a discussion about the Casimir effect. In the presence of gravitational fields, however, the contribution of this zero point energy to the effective cosmological constant of the Universe remains one of the greatest open problems in cosmology. We refer the reader to Weinberg's review [30] on the cosmological constant problem.

For a more complete analysis, we can also investigate the spectrum of the n-momentum operator  $\mathbf{P}$ , given by the spatially integrated  $(0, i)$  components of the Energy-Momentum tensor  $T^{\mu\nu}$ :

$$\mathbf{P} = - \int d^n \mathbf{x} \pi(x) \nabla \phi(x) \quad (2.39)$$

In terms of creation and annihilation operators, the momentum operator  $\mathbf{P}$  is given by

$$\mathbf{P} = \int d^n \mathbf{k} \left( \mathbf{k} \hat{a}^\dagger(\mathbf{k}) \hat{a}(\mathbf{k}) \right) \quad (2.40)$$

which is also diagonal with eigenvalues  $\mathbf{k} \in \mathbb{R}^n$ . This shows that the spectrum of the n-momentum operator  $\mathbf{P}$  corresponds to the total momentum of a system of particles  $\sum_{i=1}^N \mathbf{k}_i$ , whose eigenstates are the Fock states  $|\mathbf{k}_1 \dots \mathbf{k}_n\rangle$ .

From this analysis we can then interpret  $|\mathbf{k}_1 \dots \mathbf{k}_n\rangle$  as being the states corresponding to a system of  $N$  non-interacting particles of momentum  $\mathbf{k}_1, \dots, \mathbf{k}_N$  and total energy  $\sum_{i=1}^N E_{\mathbf{k}_i}$ .

Given this particle interpretation to the elements of the Fock space  $\mathcal{F}^+$ , we can also interpret the action of the quantum-field operator  $\hat{\phi}(x)$  on the vacuum state  $|0\rangle$ :

$$\hat{\phi}(x) |0\rangle = \int d^n \mathbf{k} f_{\mathbf{k}}^*(x) |\mathbf{k}\rangle = \int \frac{d^n \mathbf{k}}{\sqrt{2E_{\mathbf{k}}(2\pi)^n}} e^{ik_\mu x^\mu} |\mathbf{k}\rangle \quad (2.41)$$

which is, apart from the normalization  $\sqrt{2E_{\mathbf{k}}(2\pi)^n}$ , the non-relativistic expression for the particle state localized at  $x$ . We therefore interpret the action of the field-operator  $\phi(x)$  as the creation of a particle state localized at  $x$  from the vacuum. The probability amplitude for the observation of a particle at  $x$ , given that a particle was first observed at  $y$  can, therefore, be expressed as the vacuum expectation value

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^n \mathbf{k}}{2E_{\mathbf{k}}(2\pi)^n} e^{-ik_\mu(x-y)^\mu} \quad (2.42)$$

where we assume  $x^0 > y^0$ , and the equality follows from the canonical commutation relations for  $\hat{a}$  and  $\hat{a}^\dagger$ , as well as  $\hat{a}(\mathbf{k}) |0\rangle = 0$ .

In fact, since the creation and annihilation operators can be written in terms of the field  $\hat{\phi}$

$$\hat{a}(\mathbf{k}) = \langle f_{\mathbf{k}}, \hat{\phi} \rangle_{KG} \quad \text{and} \quad \hat{a}^\dagger(\mathbf{k}) = -\langle f_{\mathbf{k}}^*, \hat{\phi} \rangle_{KG} \quad (2.43)$$

the knowledge of the n-point functions

$$G_n(x_1, \dots, x_n) = \langle 0 | \phi(x_1) \dots \phi(x_n) | 0 \rangle \quad (2.44)$$

on the vacuum allows us to determine the expectation value of any observable  $\mathcal{O}[\phi]$  on a given state  $|\Psi\rangle \in \mathcal{F}^+$  of our system<sup>11</sup>. Thus finding the exact form of the n-point functions  $G_n$ , or correspondingly the time-ordered n-point functions

$$G_n^T(x_1, \dots, x_n) = \langle 0 | \mathcal{T} [\phi(x_1) \cdots \phi(x_n)] | 0 \rangle \quad (2.45)$$

are equivalent to “solving” the theory, in the sense that the expectation values of observables on the states of the system are determined by them. For example, the time-ordered 2-point function  $G_n^T(x_1, x_2)$  is extremely important in perturbative QFT, and is usually called the Feynman propagator (normally denoted by  $i\Delta_F(x_1, x_2)$ ) for the scalar field:

$$i\Delta_F(x_1, x_2) = G_2^T(x_1, x_2) = i \int \frac{d^{1+n}p}{(2\pi)^{1+n}} \frac{e^{-ip_\mu(x_1-x_2)^\mu}}{p_\mu p^\mu - m^2 + i\epsilon} \quad (2.46)$$

For the free scalar field theory the n-point functions  $G_n$  can be determined exactly from the field expansion (2.31) in terms of the 2-point function (2.42):

$$G_{2n}(x_1, \dots, x_{2n}) = \sum_{\sigma} \prod_{i=1}^{2n-1} G_2(x_{\sigma(i)}, x_{\sigma(i+1)}) \quad (2.47)$$

where  $\sigma$  runs over all possible pairings of  $\{1, \dots, 2n\}$  with  $\sigma(i) > \sigma(j)$  for all  $i > j$ . From the solution for above<sup>12</sup> we can see that the only non-zero n-point functions are the ones corresponding to an even number of spacetime points  $G_{2n}$ , which is a characteristic of theories with no particle creation processes such as free theories.

## 2.3 Path Integral Quantization

### 2.3.1 Path Integrals in Quantum Mechanics

The canonical quantization procedure described in section 2.2 is only one of the possible ways to extend quantum mechanics to systems of fields. A different one is based on Feynman’s path-integral formulation of quantum mechanics [32, 33], where the fundamental object (instead of the operators for the theory) is now the transition amplitude between spacetime points  $x_2$  and  $x_1$ , called the Feynman kernel:

$$U(x_1, x_2) = \langle t_1, \mathbf{x}_1 | t_2, \mathbf{x}_2 \rangle = \int \mathcal{D}[\mathbf{x}(t)] \mathcal{D}[\mathbf{p}(t)] \exp \left\{ i \int_{t_2}^{t_1} dt [\mathbf{p} \dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{q})] \right\} \quad (2.48)$$

where  $\mathcal{D}[\mathbf{x}(t)]$ ,  $\mathcal{D}[\mathbf{p}(t)]$  are measures on the space of all paths<sup>13</sup> and the integration on  $\mathbf{x}(t)$  runs over all histories<sup>14</sup>  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  connecting  $x_2$  to  $x_1$ .

<sup>11</sup> This is a special case of the Wightman reconstruction theorem in axiomatic QFT, which states that the knowledge of the n-point functions of the theory is enough to recover the Hilbert space of the theory along with its quantum field operators. For more details we refer the reader to [31]

<sup>12</sup> Equation (2.47) is a particular case of Wick’s theorem that can be proven in perturbative QFT. For more details see chapter 4 of [28] or chapter 6 of [23].

<sup>13</sup> A rigorous construction of such measures in the case of quantum mechanics is available for certain class of interactions. See for example chapter 3 of [34] or chapter 2 of [35] for a discussion on the Feynman-Kac formalism.

<sup>14</sup> At the moment we are only treating the usual non-relativistic quantum mechanical path integral, and thus the integration runs over all paths defined in configuration space instead of Minkowski space.

The knowledge of the Feynman kernel  $U(x_1, x_2)$  is equivalent to finding the solutions for the Schrodinger equation in non-relativistic quantum mechanics: Given an initial condition  $\psi(t_0, \mathbf{x})$  on the wavefunction  $\psi$ , the solution for later times can be shown to be

$$\psi(t, \mathbf{x}) = \int d\mathbf{y} U((t, \mathbf{x}), (t_0, \mathbf{y})) \psi(t_0, \mathbf{y}) \quad (2.49)$$

This formalism allows one to construct the transition amplitudes  $\langle t_1, \mathbf{x}_1 | t_2, \mathbf{x}_2 \rangle$  from the classical Hamiltonian of the system alone, without having to make any explicit reference to non-commutative operators and Hilbert space of states.

Observables, such as the N-products of coordinate operators  $\mathbf{x}(t)$ , can be determined from the path integral (2.48) following

$$\begin{aligned} \langle t_A, \mathbf{x}_A | \mathcal{T} [\mathbf{x}(t_1) \cdots \mathbf{x}(t_N)] | t_B, \mathbf{x}_B \rangle = \\ \int \mathcal{D}[\mathbf{x}(t)] \mathcal{D}[\mathbf{p}(t)] \mathbf{x}(t_1) \cdots \mathbf{x}(t_N) \exp \left\{ i \int_{t_B}^{t_A} dt [\mathbf{p}\dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{q})] \right\} \end{aligned} \quad (2.50)$$

where  $T$  is the time order operator, which ordinates the operators inside it in descending order with respect to its time argument  $t$  (with the latest time at the utmost left). In particular, we can write the ground-state expectation value of such time-ordered N-products in the form

$$\begin{aligned} \langle 0 | \mathcal{T} [\mathbf{x}(t_1) \cdots \mathbf{x}(t_N)] | 0 \rangle = \\ Z^{-1}[0] \int \mathcal{D}[\mathbf{x}(t)] \mathcal{D}[\mathbf{p}(t)] \mathbf{x}(t_1) \cdots \mathbf{x}(t_N) \exp \left\{ i \int_{-\infty}^{\infty} dt [\mathbf{p}\dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{q})] \right\} \end{aligned} \quad (2.51)$$

where

$$Z[0] = \int \mathcal{D}[\mathbf{x}(t)] \mathcal{D}[\mathbf{p}(t)] \exp \left\{ i \int_{-\infty}^{\infty} dt [\mathbf{p}\dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{q})] \right\} \quad (2.52)$$

and the complex limiting processes  $(-\infty \rightarrow -e^{i\epsilon}\infty)$  and  $(\infty \rightarrow e^{i\epsilon}\infty)$  for convergence are understood.

By defining the generating functional (also known as vacuum-to-vacuum amplitude or vacuum persistence amplitude functional)

$$Z[J] = Z^{-1}[0] \int \mathcal{D}[\mathbf{x}(t)] \mathcal{D}[\mathbf{p}(t)] \exp \left\{ i \int_{-\infty}^{\infty} dt [\mathbf{p}\dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{q}) + J(t)\mathbf{x}] \right\} \quad (2.53)$$

we can write the time-ordered ground-state expectation value of  $\mathbf{x}(t_1) \cdots \mathbf{x}(t_N)$  as the functional derivative

$$\langle 0 | \mathcal{T} [\mathbf{x}(t_1) \cdots \mathbf{x}(t_N)] | 0 \rangle = (-i)^N \frac{\delta^N Z[J]}{\delta J(t_1) \cdots \delta J(t_N)} \Big|_{J=0} \quad (2.54)$$

with respect to  $Z[J]$ . Thus the knowledge of the generating functional of the theory is equivalent to the knowledge of its quantum-mechanical time-ordered N-point functions  $G_N^T(t_1, \dots, t_N)$ .

### 2.3.2 Path Integrals in QFT

The generalization of the path-integral formulation of quantum mechanics to field systems is straightforward. The degrees of freedom of the theory are now field configurations  $\phi(x)$ , and the path-integral for such systems now sum over all available classical configurations of the fields. We can thus write all expressions of the previous section by making the changes

$$\begin{aligned}\mathbf{x}(t), \mathbf{p}(t) &\longrightarrow \phi(x), \pi(x) \\ \int \mathcal{D}[\mathbf{x}(t)] \mathcal{D}[\mathbf{p}(t)] &\longrightarrow \int \mathcal{D}[\phi(x)] \mathcal{D}[\pi(x)]\end{aligned}$$

The generating functional  $Z[J]$  for fields can be written as

$$Z[J] = Z^{-1}[0] \int \mathcal{D}[\phi(x)] \mathcal{D}[\pi(x)] \exp \left\{ i \int d^{1+n}x \left[ \pi(x) \dot{\phi}(x) - \mathcal{H}[\phi, \pi](x) + J(x) \phi(x) \right] \right\} \quad (2.55)$$

where  $\mathcal{H}[\phi, \pi](x)$  is the Hamiltonian density of the field and the normalization  $Z[0]$  is given by

$$Z[0] = \int \mathcal{D}[\phi(x)] \mathcal{D}[\pi(x)] \exp \left\{ i \int d^{1+n}x \left[ \pi(x) \dot{\phi}(x) - \mathcal{H}[\phi, \pi](x) \right] \right\} \quad (2.56)$$

The corresponding time-ordered N-point functions  $G_N^T(x_1, \dots, x_N)$  are determined by the functional derivatives of the generating functional:

$$G_N^T(x_1, \dots, x_N) = \langle 0 | \mathcal{T} [\phi(x_1) \cdots \phi(x_N)] | 0 \rangle = (-i)^N \frac{\delta^N Z[J]}{\delta J(x_1) \cdots \delta J(x_N)} \Big|_{J=0} \quad (2.57)$$

and thus we can find all N-point functions by calculating the path-integral for  $Z[J]$  as an explicit functional of the sources  $J(x)$ .

For theories whose Hamiltonian density takes the form

$$\mathcal{H}[\phi, \pi] = \mathcal{H}_{Kin}[\pi] + \mathcal{H}_{Non-Kin}[\phi] \quad (2.58)$$

where the  $\pi(x)$ -dependence of the Hamiltonian density is quadratic and entirely contained in the kinetic term  $\mathcal{H}_{Kin}[\pi]$ , the integration in the momentum  $\pi$  can be done independently of  $\phi$  and the path-integral simplifies to

$$\int \mathcal{D}[\phi(x)] \mathcal{D}[\pi(x)] F[\phi, \pi] \rightarrow \mathcal{N} \int \mathcal{D}[\phi(x)] F[\phi] \quad (2.59)$$

where  $\mathcal{N}$  is a normalization constant that does not affect any physical observable, since we normalized the path-integrals by the  $Z^{-1}[0]$  term which cancels such constants. The simplified expression for the generating functional of theories with such  $\pi$ -dependence is given by

$$Z[J] = Z^{-1}[0] \int \mathcal{D}[\phi(x)] \exp \left\{ i \int d^{1+n}x \left( \mathcal{L}[\phi, \partial_\mu \phi](x) + J(x) \phi(x) \right) \right\} \quad (2.60)$$



### 2.3.3 Scalar Field Quantization

We shall now proceed to find the generating functional  $Z[J]$  for the scalar field (2.9). Since the scalar field  $\pi$ -dependence is quadratic and entirely contained in the kinetic term of the Lagrangian density, to find the generating functional we must integrate (2.60) for the scalar field:

$$Z[J] = Z^{-1}[0] \int \mathcal{D}[\phi(x)] e^{i \int d^{1+n}x \left( \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + J(x) \phi(x) \right)} \quad (2.61)$$

To do this, we must find a way to relate equation (2.61), whose exponential weight is purely imaginary and thus oscillatory for “large”  $\phi$ , to the exactly-solvable, exponentially suppressed gaussian integral

$$\int \mathcal{D}[\phi(x)] e^{-\frac{1}{2}(\phi, A\phi) + (\rho, \phi)} = (\det A)^{-\frac{1}{2}} e^{\frac{1}{2}(\rho, A^{-1}\rho)} \quad (2.62)$$

where the scalar product  $(\cdot, \cdot)$  is defined as

$$(\alpha, \beta) = \int d^{1+n}x \alpha(x) \beta(x) \quad (2.63)$$

and the inverse  $A^{-1}$  is defined as the operator that satisfies

$$\int d^{1+n}x A^{-1}(x', x) A(x, x'') = \delta^{1+n}(x' - x'') \quad (2.64)$$

To transform equation (2.61) into equation (2.62), we introduce the Euclidean coordinates

$$x_E^\mu = (it, \mathbf{x}) \quad (2.65)$$

for which the path-integral takes the form

$$Z[J] = Z^{-1}[0] \int \mathcal{D}[\phi(x)] e^{-\int d^{1+n}x_E \left( \frac{1}{2} \partial_\mu^E \phi \partial_\mu^E \phi + \frac{1}{2} m^2 \phi^2 - J\phi \right)} \quad (2.66)$$

Thus, in Euclidean coordinates  $x_E^\mu$ , the path-integral for the scalar field takes the form of the Gaussian integ (2.62) with

$$A(x_E, x'_E) = \left( \partial_\mu^E \partial_\mu^E + m^2 \right) \delta^{1+n}(x_E - x'_E) \quad (2.67)$$

$$\rho(x_E) = J(x_E) \quad (2.68)$$

and therefore the solution for the Euclidian path-integral  $Z_E[J]$  is given by

$$Z_E[J] = \exp \left\{ \frac{1}{2} \int d^{1+n}x_E d^{1+n}x'_E J(x'_E) A^{-1}(x'_E, x_E) J(x_E) \right\} \quad (2.69)$$

where we have used the fact that  $Z_E[0] \propto (\det A)^{-\frac{1}{2}}$  cancels out the determinant of A in the integral<sup>15</sup>. The inverse of the operator  $\left( \partial_\mu^E \partial_\mu^E + m^2 \right) \delta^{1+n}(x_E - x'_E)$  can be found by

<sup>15</sup> This determinant is a divergent quantity, and this cancellation is analogous to the scaling of the zero-point energy in the canonical quantization formalism.

solving (2.64):

$$A^{-1}(x'_E, x_E) = \int \frac{d^{1+n}p_E}{(2\pi)^{1+n}} \frac{1}{p_E^2 + m^2} \exp[-ip_E \cdot (x'_E - x_E)] \quad (2.70)$$

Switching back to Minkowski coordinates  $x^\mu = (t, \mathbf{x}) = (-it_E, \mathbf{x}_E)$ , we find that

$$\begin{aligned} A^{-1}(x', x) &= \int \frac{d^{1+n}p_E}{(2\pi)^{1+n}} \frac{1}{p_E^2 + m^2} \exp[-i(ip_0^E)(x'_0 - x_0) + \mathbf{p}_E(\mathbf{x}' - \mathbf{x})] \\ &= i \int \frac{d^{1+n}p}{(2\pi)^{1+n}} \frac{1}{p_\mu p^\mu - m^2} \exp[-ip_0(x'_0 - x_0) + \mathbf{p}(\mathbf{x}' - \mathbf{x})] \\ &= i \int \frac{d^{1+n}p}{(2\pi)^{1+n}} \frac{e^{-ip_\mu(x'-x)^\mu}}{p_\mu p^\mu - m^2} \end{aligned} \quad (2.71)$$

where in the last line we have put the expression in the usual Minkowski momentum  $p^\mu = (p^0, \mathbf{p}) = (ip_0^E, \mathbf{p}_E)$ .

Note that the  $p_0$  integration in equation (2.71) runs from  $-i\infty$  to  $+i\infty$ . In order to deform the contour to the real axis, we must circumvent the poles  $p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}$  by adding a small negative imaginary contribution to the mass

$$m^2 \rightarrow m^2 - i\epsilon \quad (2.72)$$

This contribution fixes the oscillatory behavior of (2.61) in the limit of “large”  $\phi$ , since the exponential weight in the path-integral picks up an exponentially suppressing factor  $e^{-\epsilon(\dots)}$ .

The inverse of the Klein-Gordon operator,  $A^{-1}$ , is usually written as  $i\Delta_F$ , and is the Feynman propagator (2.46) of the theory. The full solution for the generating functional of the scalar field is then given by

$$Z[J] = \exp\left[-\frac{1}{2}i \int d^{1+n}x d^{1+n}x' J(x') \Delta_F(x' - x) J(x)\right] \quad (2.73)$$

From the functional form of the generating functional above we can calculate any time-ordered N-point functions by applying (2.57). For example, the time-ordered 2-point function  $G_2^T(x_1, x_2)$  is given by

$$G_2^T(x_1, x_2) = (-i)^2 \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} = i\Delta_F(x_1, x_2) \quad (2.74)$$

which establishes that the Feynman propagator  $i\Delta_F$  is indeed the time-ordered amplitude of a particle propagating between spacetime points  $x_1$  and  $x_2$ .

## 2.4 Finite Temperature QFT

In this section we shall briefly discuss the topic of quantum fields at finite temperature. This topic is immense, being of vital importance in the description of extended systems,

where the usual QFT study of pure states is no longer a good formalism. That being said, we shall not present the topic from scratch, as we did with the zero-temperature QFT case above, but only highlight the main topics in the subject, referring the reader to either [36] or [37] for a more complete treatment of the subject. Instead, we shall be interested in presenting a particular formalism of finite-temperature QFT, namely the thermo-field dynamical formalism. This formalism has the advantage of being able to extend methods from the usual zero-temperature formalism to systems with finite-temperature. This characteristic will prove useful when studying QFT from the point of view of accelerated observers later on.

### 2.4.1 Fields and Temperature

Finite temperature quantum field theory was developed in order to describe bulk properties of field systems near the equilibrium. In this formalism we are interested in describing the statistical behavior of the system, not on the particular behavior of each available degree of freedom.

In order to do this it is usually introduced the concept of an ensemble, which is an idealization of a set of all possible configurations of the system with prescribed bulk properties. Several ensembles can be introduced, such as the microcanonical and the canonical ensemble, but for applications in QFT, where particle-creating processes are allowed, the most common ensemble to use is the grand canonical ensemble, whose partition function can be expressed as

$$Z(V, T, \mu_i) = \text{Tr} [\exp(-\beta \mathcal{H})] \quad (2.75)$$

where the grand canonical weight  $\mathcal{H}$  is given by

$$\mathcal{H} = H - \mu_i N_i \quad (2.76)$$

with  $H$  being the total Hamiltonian for the system and  $\mu_i$  is the chemical potential for each species of particles  $N_i$ . In this thesis we shall only be interested in the particular case of zero-chemical potential  $\mu_i = 0$ , but in this chapter we will keep using  $\mathcal{H}$  for completeness.

From the partition function (2.75) we can calculate statistical averages of observables in the ensemble:

$$\langle O \rangle = Z^{-1} \text{Tr} [O e^{-\beta \mathcal{H}}] \quad (2.77)$$

and the thermodynamical properties of the system:

$$P = \frac{\partial}{\partial V} (T \ln Z) \quad (2.78)$$

$$N_i = \frac{\partial}{\partial \mu_i} (T \ln Z) \quad (2.79)$$

$$S = \frac{\partial}{\partial T} (T \ln Z) \quad (2.80)$$

$$E = -PV + TS + \mu_i N_i \quad (2.81)$$

Thus, the knowledge of the grand partition function  $Z$  is equivalent to the knowledge of all statistical properties of the system, in a way analogous to the zero-temperature generating functional  $Z[J]$ .

A path-integral representation of the partition function  $Z(\beta) \equiv Z[V, T, \mu_i]$  is readily available, since the trace over all states can be written as an integral over field configurations [36]:

$$Z(\beta) = \int \mathcal{D}[\pi] \int_{\text{Periodic}} \mathcal{D}[\phi] \exp \left[ \int_0^\beta d\tau \int d^3x (i\pi \partial_\tau \phi - \mathcal{H}) \right] \quad (2.82)$$

where  $\beta = 1/T$  and the integral in  $\phi$  is taken over all fields  $\beta$ -periodic (or anti-periodic for fermions) in  $\tau$  time. That means the integral runs over all field configurations that satisfy  $\phi(0, x) = \phi(\beta, x)$ , and is a direct consequence of the trace in  $Z(\beta)$ . The calculation of observables and thermodynamical quantities can thus be reduced to the problem of calculating path-integrals of this type.

### 2.4.2 Thermo Field Dynamics

A different way of approaching finite temperature systems, due to Takahashi and Umezawa [38], is to try and express the statistical averages (2.77) as a “vacuum” expectation value in an enlarged Hilbert space of states:

$$\langle O \rangle = Z^{-1}(\beta) \text{Tr} [O e^{-\beta \mathcal{H}}] = \langle 0(\beta) | O | 0(\beta) \rangle \quad (2.83)$$

where this vacuum  $|0(\beta)\rangle$  of the enlarged Hilbert space is now temperature dependent.

Takahashi and Umezawa showed that it is possible to construct a representation of the canonical commutation relations for the system in which the above equation is true. To do this, they proposed the enlargement of the Hilbert-space of the theory

$$\mathcal{H} \rightarrow \mathcal{H} \otimes \tilde{\mathcal{H}} \quad (2.84)$$

by adding a fictitious new system  $\tilde{\mathcal{H}}$ , identical to the original one, in such a way that the temperature-dependent vacuum can be expressed as

$$|0(\beta)\rangle = Z^{-1/2}(\beta) \sum_n e^{-\beta E_n/2} |n\rangle \otimes |\tilde{n}\rangle \quad (2.85)$$

where  $|n\rangle$  corresponds to the eigenstates of the Hamiltonian for the real particles in  $\mathcal{H}$ , and  $|\tilde{n}\rangle$  are the eigenstates for the Hamiltonian in  $\tilde{\mathcal{H}}$ . In that way, observables  $O_{\text{Phys}} \equiv O \otimes I_{\tilde{\mathcal{H}}}$

corresponding to the physical sector of the theory have vacuum expectation values

$$\begin{aligned}\langle O_{Phys} \rangle &= Z^{-1}(\beta) \sum_{n,m} e^{-\beta(E_n+E_m)/2} \langle n, \tilde{n} | O_{Phys} | n, \tilde{n} \rangle = \\ &= Z^{-1}(\beta) \sum_n e^{-\beta E_n} \langle n | O | n \rangle \equiv \langle O \rangle\end{aligned}\quad (2.86)$$

which are the statistical averages for a system at finite temperature. Thus we can transform a problem of finite-temperature to a zero-temperature one defined in a higher Hilbert space of states.

**Example 2.4.1** (Ensemble of Free Bosons with Frequency  $\omega$ ). *As an example of this construction, we can treat the system consisting of a single bosonic degree of freedom  $\{a, a^\dagger\}$ , whose Hamiltonian is given by*

$$H = \omega a^\dagger a \quad (2.87)$$

and the operators satisfy  $[a, a^\dagger] = 1$ . The Fock space of states of this system is spanned by  $\frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle$ , and the eigenvalues of the Hamiltonian are of the form  $n\omega$ , where  $n \in \mathbb{Z}_+$ .

To study this system at finite temperature  $\beta$ , we proceed by constructing the  $|0(\beta)\rangle$  in the enlarged space  $\mathcal{H} \otimes \tilde{\mathcal{H}}$ :

$$\begin{aligned}|0(\beta)\rangle &= Z^{-1}(\beta) \sum_n e^{-\beta n\omega/2} \frac{1}{n!} (a^\dagger)^n (\tilde{a}^\dagger)^n |0, \tilde{0}\rangle = \\ &= \frac{e^{-\beta\omega/2}}{\sqrt{1 + \gamma_B(\omega)}} \exp \left[ e^{-\beta\omega/2} a^\dagger \tilde{a}^\dagger \right] |0, \tilde{0}\rangle\end{aligned}\quad (2.88)$$

where we have written the last line in terms of the bosonic thermal factor  $\gamma_B(\omega) = (e^{\beta\omega} - 1)^{-1}$ .

By defining the temperature-dependent creation and annihilation operators

$$a(\beta) = \sqrt{1 + \gamma_B(\omega)} a - \sqrt{\gamma_B(\omega)} \tilde{a}^\dagger \quad (2.89)$$

$$\tilde{a}(\beta) = \sqrt{1 + \gamma_B(\omega)} \tilde{a} - \sqrt{\gamma_B(\omega)} a^\dagger \quad (2.90)$$

we can easily check that  $|0(\beta)\rangle$  is a vacuum state with respect to  $a(\beta), \tilde{a}(\beta)$ . Thus the explanation of the terminology “vacuum”. We call the Fock states spanned by the action of  $a^\dagger(\beta)$  on the thermal vacuum  $|0(\beta)\rangle$  the physical TFD-excitations, and  $a(\beta)$  the fictitious TFD-excitations. This denomination comes from the properties

$$\begin{aligned}a^\dagger(\beta) |0(\beta)\rangle &= \frac{1}{\sqrt{1 + \gamma_B(\omega)}} a^\dagger |0(\beta)\rangle \\ \tilde{a}^\dagger(\beta) |0(\beta)\rangle &= \frac{1}{\sqrt{\gamma_B(\omega)}} a |0(\beta)\rangle\end{aligned}\quad (2.91)$$

which specify the action of  $a^\dagger(\beta)$  and  $\tilde{a}^\dagger(\beta)$  as respectively the creation and annihilation of a physical excitation from the thermal vacuum  $|0(\beta)\rangle$ .

The expectation value of the number operator  $N_{Phys}(\omega) = (a^\dagger a) \otimes I_{\mathcal{H}}$  for physical particle excitations of energy  $\omega$  in a bosonic system at temperature  $\beta$  is then given by:

$$\langle 0(\beta) | N_{Phys} | 0(\beta) \rangle = \gamma_B(\omega) = \frac{1}{e^{\beta\omega} - 1} \quad (2.92)$$

which is the known result for free system of bosons

### 2.4.3 Path-Integral Formulation of TFD

A path-integral approach of thermo-field dynamics applied to quantum field theory, due to Niemi and Semenoff [39, 40], can also be constructed. The idea is to find a way to calculate ensemble averages of time-ordered products of a given field  $\phi$  in the same form as in zero-temperature QFT:

$$G_\beta(x_1, \dots, x_n) \equiv \langle \mathcal{T}(\phi(x_1), \dots, \phi(x_n)) \rangle = \frac{1}{Z_\beta[0]} (-i)^n \frac{\delta^n Z_\beta[J]}{\delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0} \quad (2.93)$$

where  $Z_\beta[J]$  is the finite-temperature equivalent of the zero-temperature generation functional. They have showed that it is possible to construct such temperature-dependent generation functional for a finite-temperature quantum field. Such functional is given by

$$Z_\beta[J] = \int_{\tau \in \mathcal{C}} \mathcal{D}[\phi] \mathcal{D}[\pi] \exp \left[ i \int_{\mathcal{C}} d^4x \left( \pi \dot{\phi} - H + J\phi \right) \right] \quad (2.94)$$

where  $\mathcal{C}$  is a path restricted to the lower complex  $t$ -plane that connects  $\tau = -T$  to  $\tau = -T - i\beta$  and passes through the points which appear through the argument of the time-ordered  $n$ -point function one wishes to determine. This is a generalization of path-ordering, and the integral on  $\phi$  is naturally constrained by the periodic boundary condition  $\phi(-T) = \phi(-T - i\beta)$ . One possible path of this kind is showed on figure 1. The freedom in the choice of such paths allows us to construct the most convenient possibility for each individual case, as we do later in the case of the Unruh effect.

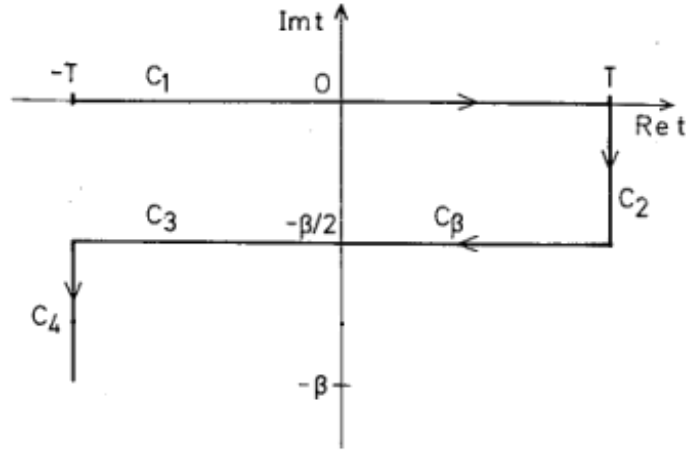
The time-ordered  $n$ -point functions for a quantum field at finite temperature can thus be written as the path-integral over the complex contour

$$\langle T[\phi(x_1) \cdots \phi(x_n)] \rangle = \int \mathcal{D}[\phi] \phi(x_1) \cdots \phi(x_n) \exp[-S_{\mathcal{C}}[\phi]] \quad (2.95)$$

where  $S_{\mathcal{C}}[\phi]$  is the action for the theory in on the contour  $\mathcal{C}$ . In general strokes, we can also construct the thermal propagator  $D_\beta(x, y)$  for a given theory by finding the solutions for the Green's function on the contour  $\mathcal{C}$ :

$$\mathcal{W}_{\mathcal{C}} D_\beta(x, y) = \delta(x - y) \quad (2.96)$$

where  $\mathcal{W}_{\mathcal{C}}$  is the field-equation operator for the specific theory along  $\mathcal{C}$  and the delta is defined along the contour of integration chosen.

Figure 1 – Possible choice of complex contour  $\mathcal{C}$ .(Source: [16])

## 2.5 QFT in General Coordinate Systems

Although the general structure of classical field theory outlined in section 2.1.1 is valid for a broad class of (possibly curved) spacetimes  $(\mathcal{M}, g)$ , for more general cases there is no longer the guarantee of global Poincaré symmetry. In such cases, Lorentz invariance is only expected to hold locally, in the sense that for each point  $p \in \mathcal{M}$  we can always find a neighborhood  $U_p \subset \mathcal{M}$  and a coordinate system  $y_p : U_p \rightarrow V \subset \mathbb{R}^{1+n}$  such that the metric  $g^{\mu\nu}$  in terms of  $y_p$  coordinates is simply the Minkowskian metric  $\eta^{\mu\nu}$ , and on  $U_p$  the theory is Lorentz invariant.

To treat these cases, one must construct the Lagrangian density of the theory in a locally Lorentz invariant and coordinate independent way. While intuitive, these generalizations heavily influence the particle interpretation of the theory, delineated in section 2.2.2. In flat space, the vacuum state of the system depends on the existence of time translation symmetry – generated by the Killing vector  $\partial_t$  – and therefore the interpretation of particle excitations is connected with the observer’s notion of time translation. Poincaré invariance ensures that all inertial observers agree on such choice of vacuum state, since such choice is based on the Poincaré invariant concept of positive and negative frequencies with respect to the inertial timelike coordinate.

For more general cases, such as curved spacetimes and flat space associated with different timelike Killing vectors, there is no unambiguous choice of vacuum state, and therefore the concept of particles are not well defined. For particular choices of spacetime manifolds such interpretation might be applicable, but we shall not dwell on the “meaning” of particles in those cases, keeping it a matter of terminology.

### 2.5.1 Scalar Field on Curved Spacetime

As an example, we shall outline the theory of scalar fields in a general spacetime  $(\mathcal{M}, g)$ . The covariant generalization of the scalar field (2.9) to such spacetime is given by:

$$\mathcal{L}(x) = \frac{1}{2} \sqrt{|g(x)|} \left( g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - [m^2 + \xi R(x)] \phi(x) \right) \quad (2.97)$$

where  $g = \det g_{\mu\nu}$  and we have included the coupling term  $\xi R(x)$  of the field with the spacetime curvature for completeness, even though we will only treat the minimally coupled case  $\xi = 0$ . We follow the convention of linking the spacetime volume term  $\sqrt{|g(x)|}$  to the Lagrangian density instead of the action integral.

From here the canonical quantization procedure of the theory follows closely its Minkowski counterpart. One defines the scalar product

$$\langle \phi_1, \phi_2 \rangle = -i \int_\Sigma d\Sigma \sqrt{|g_\Sigma(x)|} n^\mu (\phi_1(x) \partial_\mu \phi_2^*(x) - \phi_2^*(x) \partial_\mu \phi_1(x)) \quad (2.98)$$

with  $n^\mu$  a timelike future directed unit vector orthogonal to the spacelike hypersurface  $\Sigma$ . It can be shown that this scalar product does not depend on the choice of hypersurface  $\Sigma$  [41]. We then proceed to search for orthonormal solutions  $h_p$  of the field equation

$$\left( g^{\mu\nu} \partial_\mu \partial_\nu + (m^2 + \xi R(x)) \right) \phi(x) = 0 \quad (2.99)$$

and expand the field in terms of the complete set  $\{h_p, h_p^*\}$  and the creation and annihilation operators  $\{a(p), a^\dagger(p)\}$ :

$$\phi(x) = \int d^n p \left( a(p) h_p(x) + a^\dagger(p) h_p^*(x) \right) \quad (2.100)$$

The vacuum is defined in an analogous manner to the flat-space case

$$a(p) |0_a\rangle = 0 \quad \text{for all } p \quad (2.101)$$

We can see clearly that, unless we choose a specific spacetime, there is no preferred class of modes  $\{h_p, h_p^*\}$  and no natural decomposition of the field is available. Given another choice of modes  $\{s_q, s_q^*\}$  one has another set of creation and annihilation operators  $\{b(q), b^*(q)\}$ , and consequently another vacuum  $b(q) |0_b\rangle$ . Contrary to the flat space case, it is not true that these vacuum  $|0_a\rangle$  and  $|0_b\rangle$  are equivalent in the sense that they are associated with different kinds of excitations. These excitations are connected with the choices of modes in which one expands the field, and any kind of particle interpretation would depend on such choices.

For concreteness, let us write the field  $\phi$  in terms of both  $\{a(p), a^\dagger(p)\}$  and  $\{b(q), b^\dagger(q)\}$ :

$$\phi(x) = \int d^n p \left( a(p) h_p(x) + a^\dagger(p) h_p^*(x) \right) \quad (2.102)$$

$$\phi(x) = \int d^n q \left( b(q) s_q(x) + b^\dagger(q) s_q^*(x) \right) \quad (2.103)$$



From the orthonormality of the modes with respect to the scalar product (2.98) we can write the set  $\{h_q, h_q^*\}$  in terms of  $\{s_q, s_q^*\}$  and vice versa:

$$s_q(x) = \int d^n p \left( \alpha(q, p) h_p(x) + \beta(q, p) h_p^*(x) \right) \quad (2.104)$$

$$h_p(x) = \int d^n q \left( \alpha^*(q, p) s_q(x) - \beta(q, p) s_q^*(x) \right) \quad (2.105)$$

where the coefficients are given by

$$\alpha(q, p) = \langle s_q, h_p \rangle \quad \text{and} \quad \beta(q, p) = -\langle s_q, h_p^* \rangle \quad (2.106)$$

With these expressions we can find the relation between creation and annihilation operators sets:

$$b(q) = \int d^n p \left( \alpha^*(q, p) a(p) - \beta^*(q, p) a^\dagger(p) \right) \quad (2.107)$$

$$a(p) = \int d^n q \left( \alpha(q, p) b(q) + \beta^*(q, p) b^\dagger(q) \right) \quad (2.108)$$

These coefficients  $\alpha$  and  $\beta$  are called Bogoliubov transformation coefficients, and they encode the information about how the choices of modes  $\{h_p, h_p^*\}$  and  $\{s_q, s_q^*\}$  and their respective creation/annihilation operators relate to each other.

From equation (2.107) we can see that if the chosen modes satisfy  $\beta(q, p) > 0$  we have:

$$\langle 0_a | b^\dagger(q) b(q) | 0_a \rangle = \int d^n p |\beta(q, p)|^2 > 0 \quad (2.109)$$

which demonstrate that in fact the vacua  $|0_a\rangle$  and  $|0_b\rangle$  are associated with different and inequivalent notions of “particles”.

## Chapter 3

# The Unruh Effect

In chapter 2 we briefly discussed the peculiarities of quantum field theory in curved spacetime. In particular, the elusive nature of the particle concept. As we shall discuss in this chapter, this feature is also present in flat space when dealing with non-inertial coordinate systems. The Unruh effect is a reflection of that feature, stating that an uniformly accelerated observer perceives the inertial vacuum as a thermal bath of particles in a temperature proportional to its acceleration. The explanation is exactly the same to the curved spacetime case: the inequivalent concept of particles between the inertial and accelerated observers. For simplicity of presentation we discuss here, without loss of generality, the (1+1) dimensional free scalar field, and a brief outline of the general demonstration for interacting theories is presented at the end.

### 3.1 Preliminaries

The Unruh effect is a statement about inertial and accelerated observers, and whenever we are dealing with two distinct points of view special care is needed in order to avoid confusion. Before discussing the Unruh effect *per se*, we must first make precise some of the terminology that we will be using.

We will identify an observer  $\mathcal{O}$  as a timelike worldline  $O : \mathbb{R} \rightarrow \mathcal{M}$ , defined on the flat spacetime  $\mathcal{M}$ , which is capable of performing local measurements on the physical system in question. Observers so defined are characterized by their state of motion, and therefore independent of coordinate systems. A reference frame associated with the observer is a coordinate system  $(x^0, x^i)_{\mathcal{O}}$  in which the observer's worldline  $O$  appears to be at rest, and therefore can be described by  $x^i(x^0) = \text{const}$  for all  $i$ . This means that we can always put the observer in the origin of the coordinate system and speak freely about the observer corresponding to a given reference frame. We note that it is not always possible to construct global coordinates  $(x^0, x^i)_{\mathcal{O}}$  for a given observer<sup>1</sup>.

---

<sup>1</sup> An accelerated observer – as we will discuss in this chapter – is one of such cases, where the existence of

Based on this definitions we can now establish the reference frames relevant to the study of the Unruh effect:

- The laboratory frame – an inertial observer together with standard coordinates  $(t, x) \in \mathcal{M}^2$  on Minkowski space. We will often refer to the laboratory frame as the inertial (or Minkowski) frame
- The accelerated frame, which is the reference frame associated with the observer whose worldline has constant proper acceleration. We shall refer to this reference frame as either accelerated or Rindler frame.

Given such definitions, we now proceed to specify the accelerated coordinate system associated with the accelerated observer.

## 3.2 Accelerated Observer

The worldline of an uniformly accelerated observer is characterized by the condition of constant proper acceleration  $a \in \mathbb{R}$  in the instantaneously comoving reference frame. This condition can be translated to the relativistically invariant relation

$$\ddot{x}_\mu \ddot{x}^\mu = -a^2 \quad (3.1)$$

where the dot represents the derivative with respect to the proper time  $\tau$ . This equation can be solved exactly[42], and together with suitable boundary conditions the solution can be parametrized as

$$x(\tau) = a^{-1} \cosh a\tau \quad \text{and} \quad t(\tau) = a^{-1} \sinh a\tau \quad (3.2)$$

which are branches of hyperboles whose asymptotes are the lightlike boundaries of the lightcone  $x \pm t = 0$ . This is expected, since no trajectory can have constant acceleration with respect to an inertial reference frame.

From this worldline we can write a coordinate system for the the accelerated observer:

$$x = a^{-1} e^{a\xi} \cosh a\tau \quad t = a^{-1} e^{a\xi} \sinh a\tau \quad \xi \in \mathbb{R} \quad \tau \in \mathbb{R} \quad (3.3)$$

where  $\tau$  and  $\xi$  are timelike and spacelike coordinates respectively.

These coordinates are usually called conformal Rindler coordinates. The simultaneity hypersurfaces associated with this accelerated observer are straight lines passing through the origin, corresponding to  $\tau = \text{const}$ . The spacelike coordinate  $\xi$  corresponds to hyperboles associated with different accelerations. The accelerated observer follows the worldline  $\xi(\tau) = 0$  in  $(\tau, \xi)$  coordinates.

---

causally inaccessible regions of Minkowski spacetime prevents the construction of such global coordinate chart.

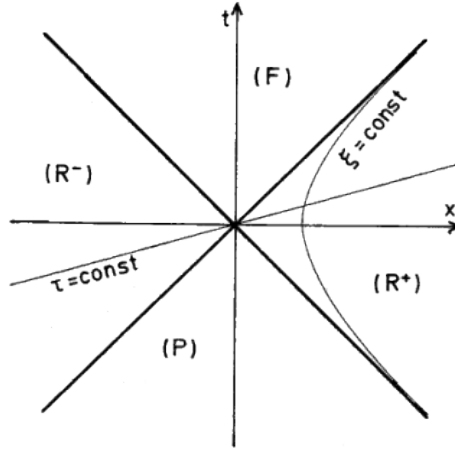


Figure 2 – Wedge diagram for Minkowski spacetime. The right wedge  $R_+$  is the region where the observer (here indicated by the hyperbola  $\xi = \text{const}$ ) is confined. The future and past wedges  $F, P$  are the regions where  $R_+$ -communication is limited. The left wedge  $R_-$  is the causally inaccessible region. (Source: [16])

An important characteristic of the accelerated reference frame (3.3) is the fact that it does not cover the entire Minkowski space  $(t, x) \in \mathcal{M}^2$ . This is due to fact that the worldline (3.2) is entirely contained in the region  $x > |t|$  of the Minkowski space, called the right Rindler wedge  $R$ . This region cannot send signals to the Past region  $P$ , defined by  $t < |x|$ , receive signals from the Future region  $F$ , defined by  $t > |x|$ , and is completely causally disconnected from the left Rindler wedge  $L$ , defined by  $x < |t|$ . Observers whose worldline are entirely contained in the right Rindler wedge cannot therefore assign coordinates to any of these regions other then  $R$  itself, and the asymptotes  $x \pm t = 0$  act as event horizons for the accelerated observer.

Since the coordinate transformation (3.3) is not a Poincaré transformation, it does not preserve the Minkowski line element. The line element and metric tensor associated with such coordinate transformation is given by

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) = g_R^{\mu\nu} d\xi^\mu d\xi^\nu \quad g_R^{\mu\nu}(\eta, \xi) = e^{2a\xi} \eta^{\mu\nu} \quad (3.4)$$

We can consider the right Rindler wedge together with the metric tensor  $g_R^{\mu\nu}$  as a spacetime of its own, called Rindler space. This metric is independent of the timelike coordinate  $\tau$ , and therefore Rindler space is a static spacetime. We note that since the reference frame (3.3) is only an unusual set of coordinates for the right Rindler wedge region of the Minkowski space, the intrinsic curvature of Rindler space is still zero.

### 3.3 Free Fields in Rindler Space

An important symmetry of the Rindler space is the proper-time translation  $\tau \rightarrow \tau + \tau_0$ , whose associated Killing vector is  $\partial_\tau$ . This is a reflection of the boost invariance of the

Minkowski space, since Lorentz boosts along the acceleration axis  $x^\mu \rightarrow (\Lambda_{Boost})^\mu_\nu x^\nu$  act as proper-time translations in Rindler space. The existence of timelike Killing vectors, as we saw in section 2.5, allows us to define the positive and negative frequency modes that are fundamental in the physical interpretation of particles.

### 3.3.1 Scalar Field

Even though massless scalar fields are problematic in (1+1) spacetime dimensions due to irredeemable infrared divergences<sup>2</sup>, they are the cleanest way to present the Unruh effect. Since such divergences does not impact the conclusion, we shall choose the clearer approach and work with massless fields.

Free massless scalar fields in (1+1) spacetime dimensions are represented by the Klein-Gordon Lagrangian (2.9) with  $m = 0$ :

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) = \frac{1}{2} \left( (\partial_t \phi(t, x))^2 - (\partial_x \phi(t, x))^2 \right) \quad (3.5)$$

For convenience, we rewrite the solution for the massless field  $\phi$  for the particular case of (1+1) dimensions:

#### Field Expansion

$$\phi(t, x) = \int dk \left( a(k) f_k(t, x) + a^\dagger(k) f_k^*(t, x) \right) \quad (3.6)$$

#### Positive Frequency Modes

$$f_k(t, x) = \frac{e^{ikx - iE_k t}}{\sqrt{4\pi E_k}}, \quad E_k = |k| \quad \text{and} \quad k \in \mathbb{R} \quad (3.7)$$

#### Commutation Relations

$$[a(k), a^\dagger(k')] = \delta(k - k') \quad [a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \quad (3.8)$$

#### Vacuum State

$$a(k) |0_M\rangle = 0 \quad \text{for all } k \in \mathbb{R} \quad (3.9)$$

This information, as we've discussed in chapter 1, allows us to calculate any desired observable for the free theory. We will refer to this solution as the Minkowski (or inertial) picture, and the vacuum state  $|0_M\rangle$  associated with the operators  $a(k)$ ,  $a^\dagger(k)$  will be called

<sup>2</sup> In (1+1) spacetime dimensions the propagator for the massless scalar field is not a temperate distribution. That means that the propagator is too singular to define a meaningful quantum field theory. For more details see chapter 2 of [24].

the Minkowski vacuum.

We now consider quantum fields from the point of view of accelerated observers, whose natural timelike Killing vector  $\partial_\tau$  specify the Rindler space  $(R_+, g_R)$  as its associated spacetime.

This Lagrangian for the field  $\phi$  in Rindler space can be found by writing the usual Lagrangian (3.5) in terms of the conformal Rindler coordinates (3.3). We can do this by either direct substitution on the action or by using the covariant form (2.97):

$$\mathcal{L} = \frac{1}{2} \sqrt{|g_R|} g_R^{\mu\nu} \partial_\mu^R \phi(\xi) \partial_\nu^R \phi(\xi) = \frac{1}{2} \left( (\partial_\tau \phi(\tau, \xi))^2 - (\partial_\xi \phi(\tau, \xi))^2 \right) \quad (3.10)$$

where  $\partial_\mu^R$  refers to differentiation with respect to Rindler coordinates  $\tau, \xi$ . This Lagrangian has the same form as its inertial (3.5) counterpart<sup>3</sup>, and therefore the field equation in Rindler coordinates is also of the same form as the inertial case. Thus the solution of the Klein-Gordon equation in accelerated coordinates can be found in the same way:

#### Field Expansion

$$\phi(\tau, \xi) = \int dp \left( b(p) g_p(\tau, \xi) + b^\dagger(p) g_p^*(\tau, \xi) \right) \quad (3.11)$$

#### Positive Frequency Modes

$$g_p(\tau, \xi) = \frac{e^{ip\xi - iE_p\tau}}{\sqrt{4\pi E_p}}, \quad E_p = |p| > 0 \quad \text{and} \quad p \in \mathbb{R} \quad (3.12)$$

#### Commutation Relations

$$[b(p), b^\dagger(p')] = \delta(p - p') \quad [b(p), b(p')] = [b^\dagger(p), b^\dagger(p')] = 0 \quad (3.13)$$

#### Vacuum State

$$b(p) |0_R\rangle = 0 \quad \text{for all } p \in \mathbb{R} \quad (3.14)$$

This solution will be referred to as the Rindler (or accelerated) picture, and the vacuum state  $|0_R\rangle$  will be called the Rindler vacuum.

It is important to note that the notions of positive frequency modes changed from the Minkowski to the Rindler picture. One notion is based on the time-translation symmetry  $t \rightarrow t + t_0$  and the other on proper-time translation symmetry  $\tau \rightarrow \tau + \tau_0$ , which are intrinsically distinct.

<sup>3</sup> This is the advantage of using the conformal coordinates (3.3)

### 3.3.2 Bogoliubov Transformation

In order to show the thermal nature of the Minkowski vacuum  $|0_M\rangle$  with respect to Rindler particles, we must first find a relationship between the Rindler particle operators  $b(p), b^\dagger(p)$  and the Minkowski particle operators  $a(k), a^\dagger(k)$ . Since both Minkowski and Rindler pictures represent the same field on the right Rindler wedge of the Minkowski space, we can find such relationship by connecting the solutions with a Bogoliubov transformation, in the same way as in section 2.5, from the condition

$$\phi(t, x) = \phi(\tau, \xi) \quad \text{whenever} \quad (t, x) \in R_+ \subset \mathcal{M}^2 \quad (3.15)$$

In order to implement this condition, it is often useful to rewrite the fields in terms of the lightcone coordinates

$$u = t - x \quad v = t + x \quad (3.16)$$

and

$$\tilde{u} = \tau - \xi \quad \tilde{v} = \tau + \xi \quad (3.17)$$

In lightcone coordinates  $(u, v)$  the Rindler wedge  $R_+$  has a particularly simple form given by

$$R_+ = \{(u, v) \in \mathcal{M}^2 \mid u < 0 \text{ and } v > 0\} \quad (3.18)$$

and the fields can be separated into  $u(\tilde{u})$  and  $v(\tilde{v})$  parts as

Inertial Picture

$$\begin{aligned} \phi(u, v) &= \int_0^\infty dk \left( a(k) f_k(u) + a^\dagger(k) f_k^*(u) + a(-k) f_{-k}(v) + a^\dagger(-k) f_{-k}(v) \right) \quad (3.19) \\ f_k(u) &= \frac{e^{-iku}}{\sqrt{4\pi E_k}} \quad f_{-k}(v) = \frac{e^{-ikv}}{\sqrt{4\pi E_k}} \end{aligned}$$

Accelerated Picture

$$\begin{aligned} \phi(\tilde{u}, \tilde{v}) &= \int_0^\infty dp \left( b(p) g_p(\tilde{u}) + b^\dagger(p) g_p^*(\tilde{u}) + a(-k) g_{-p}(\tilde{v}) + b^\dagger(-p) g_{-p}(\tilde{v}) \right) \quad (3.20) \\ g_p(\tilde{u}) &= \frac{e^{-ip\tilde{u}}}{\sqrt{4\pi E_p}} \quad g_{-p}(\tilde{v}) = \frac{e^{-ip\tilde{v}}}{\sqrt{4\pi E_p}} \end{aligned}$$

From the relationship

$$u = -\frac{e^{-a\tilde{u}}}{a} \quad v = \frac{e^{a\tilde{v}}}{a} \quad (3.21)$$

between  $(u, v)$  and  $(\tilde{u}, \tilde{v})$  we see that  $u = u(\tilde{u})$  and  $v = v(\tilde{v})$ , and thus the  $u/\tilde{u}$  and  $v/\tilde{v}$  parts of (3.19) and (3.20) do not mix. We can therefore equate the  $u/\tilde{u}$  and  $v/\tilde{v}$  parts of the fields  $\phi(u, v)$  and  $\phi(\tilde{u}, \tilde{v})$  independently:

$$\int_0^\infty dk \left( a(k) f_k(u) + a^\dagger(k) f_k^*(u) \right) = \int_0^\infty dp \left( b(p) g_p(\tilde{u}) + b^\dagger(p) g_p^*(\tilde{u}) \right) \quad (3.22)$$

$$\int_0^\infty dk \left( a(-k) f_{-k}(v) + a^\dagger(-k) f_{-k}^*(v) \right) = \int_0^\infty dp \left( a(-k) g_{-p}(\tilde{v}) + b^\dagger(-p) g_{-p}^*(\tilde{v}) \right) \quad (3.23)$$

From these equations we can finally find the relationship between the Rindler operators  $b(p), b^\dagger(p)$  and the Minkowski  $a(k), a^\dagger(k)$  ones. To do this we can either use the orthonormality of the modes  $g_p, g_p^*$  or a Fourier transform to isolate the Rindler operators. We use the Fourier transform and integrate both sides of (3.22) by  $\sqrt{\frac{E_p}{\pi}} \int d\tilde{u} e^{-ip\tilde{u}}$ . The right hand side is given by:

$$\sqrt{\frac{E_p}{\pi}} \int_{-\infty}^\infty d\tilde{u} e^{ip\tilde{u}} \int_0^\infty \frac{dp'}{\sqrt{4\pi E_{p'}}} \left( b(p') e^{-ip'\tilde{u}} + b^\dagger(p') e^{ip'\tilde{u}} \right) = \begin{cases} b(p), & \text{if } p > 0 \\ b^\dagger(-p), & \text{if } p < 0 \end{cases} \quad (3.24)$$

and the right hand side is:

$$\sqrt{\frac{E_k}{\pi}} \int_{-\infty}^\infty d\tilde{u} e^{ip\tilde{u}} \int_0^\infty \left( a(k) f_k(u) + a^\dagger(k) f_k^*(u) \right) = \int_0^\infty dk \left( \alpha_{pk} a(k) + \beta_{pk} a^\dagger(k) \right) \quad (3.25)$$

where  $\alpha_{pk}$  and  $\beta_{pk}$  are the Bogoliubov transformation coefficients between the Rindler and Minkowski operators:

$$\alpha_{pk} = \frac{1}{2\pi} \sqrt{\frac{E_p}{E_k}} \int d\tilde{u} e^{ip\tilde{u} - iku(\tilde{u})} \quad \text{and} \quad \beta_{pk} = \frac{1}{2\pi} \sqrt{\frac{E_p}{E_k}} \int d\tilde{u} e^{ip\tilde{u} + iku(\tilde{u})} \quad (3.26)$$

The distributions associated with the integrals in (3.26) can be calculated exactly [43, 44]:

$$\alpha_{pk} = \theta(pk) \frac{a^{-\frac{ip}{a} - 1}}{2\pi} \Gamma\left(-\frac{ip}{a}\right) E_p^{\frac{1}{2}} E_k^{\frac{ip}{a} - \frac{1}{2}} e^{\frac{\pi p}{2a} \text{sign}(k/a)} \quad (3.27)$$

$$\beta_{pk} = \theta(pk) \frac{a^{-\frac{ip}{a} - 1}}{2\pi} \Gamma\left(-\frac{ip}{a}\right) E_p^{\frac{1}{2}} E_k^{\frac{ip}{a} - \frac{1}{2}} e^{-\frac{\pi p}{2a} \text{sign}(k/a)} \quad (3.28)$$

and expression for the Rindler annihilation operator  $b(p)$  for  $p > 0$  can thus be written in terms of the Bogoliubov coefficients as

$$b(p) = \int_0^\infty dk \left( \alpha_{pk} a(k) + \beta_{pk} a^\dagger(k) \right) \quad (p > 0) \quad (3.29)$$

The same procedure can be done for the  $v/\tilde{v}$  equality in order to get an expression for  $p < 0$ . Integrating by  $\sqrt{\frac{E_p}{\pi}} \int d\tilde{v} e^{-ip\tilde{v}}$  both sides we find that:

$$b(p) = \int_0^\infty dk \left( \alpha_{p(-k)} a(-k) + \beta_{p(-k)} a^\dagger(-k) \right) \quad (p < 0) \quad (3.30)$$



For future convenience, we assume  $a \geq 0$  and write the general expression for the  $b(p)$  operator:

$$b(p) = \frac{a^{-\frac{ip}{a}-1}}{2\pi} \int dk \theta(pk) \Gamma\left(-\frac{ip}{a}\right) E_p^{-\frac{1}{2}} E_k^{\frac{ip}{a}-\frac{1}{2}} \left( e^{\pi E_p/2a} a(k) + e^{-\pi E_p/2a} a^\dagger(k) \right) \quad (3.31)$$

where we have used the fact that  $\theta(pk) \exp\left[\frac{\pi p}{2a} \text{sign}(k/a)\right] = \theta(pk) \exp[\pi E_p/2a]$  for  $a \geq 0$ .

### 3.3.3 Rindler Particles in Minkowski Vacuum

With the expression for the Rindler particle operators  $b(p)$  in terms of Minkowski operators  $a(k), a^\dagger(k)$  we can calculate the average number of Rindler particles in the Minkowski vacuum  $|0_M\rangle$ . In order to do this we calculate the slightly more general observable

$$N(p, p') = \langle 0_M | b^\dagger(p) b(p') | 0_M \rangle \quad (3.32)$$

where we will assume, without loss of generality, positive momenta  $p, p' > 0$ .

Substituting the expression (3.29) in the right hand side of the above equation gives us an expression for  $N(p, p')$  in terms of the Bogoliubov coefficients (3.26):

$$N(p, p') = \int_0^\infty \int_0^\infty dk dk' \beta_{pk}^* \beta_{p'k'} \langle 0_M | a(k) a^\dagger(k') | 0_M \rangle = \int_0^\infty dk \beta_{pk}^* \beta_{p'k} \quad (3.33)$$

Together with the expression for the Bogoliubov coefficient  $\beta$ , we have an equation for the observable  $N(p, p')$ :

$$N(p, p') = \frac{a^{-i\frac{(p'-p)}{a}-2}}{4\pi^2} \Gamma\left(-\frac{ip'}{a}\right) \Gamma\left(\frac{ip}{a}\right) p^{1/2} p'^{1/2} e^{-\frac{\pi}{2a}(p'+p)} \int_0^\infty dk k^{-i\frac{(p'-p)}{a}-1} \quad (3.34)$$

The  $k$  integral can be solved in terms of the delta distribution by making the change of variables  $x = a^{-1} \log k$ :

$$\int_0^\infty dk k^{-i\frac{(p'-p)}{a}-1} = 2\pi a \delta(p - p') \quad (3.35)$$

and therefore the expression for  $N(p, p')$  can be written as:

$$N(p, p') = \frac{p}{2\pi a} \Gamma\left(\frac{ip}{a}\right) \Gamma\left(-\frac{ip}{a}\right) e^{-\frac{\pi p}{a}} \quad (3.36)$$

Now using the property of the Gamma function

$$\Gamma(iy)\Gamma(-iy) = \frac{\pi}{y \sinh(\pi y)} \quad (3.37)$$

we have the final expression

$$N(p, p') = \left[ \exp\left(\frac{2\pi E_p}{a}\right) - 1 \right]^{-1} \delta(p - p') \quad (p, p' > 0) \quad (3.38)$$

From the above equation we can see that the expression for the average number of (positive momenta) Rindler particles in the Minkowski vacuum is given by

$$N(p) = \left[ \exp \left( \frac{2\pi E_p}{a} \right) - 1 \right]^{-1} \delta(0) \quad (3.39)$$

where the divergent part  $\delta(0)$  is the reflection of the fact we are counting the number of particles in the entire space. The particle density is finite, and is given by the Bose-Einstein distribution

$$n_T(p) = \left[ \exp \left( \frac{2\pi E_p}{a} \right) - 1 \right]^{-1} \quad (3.40)$$

for particles at a temperature

$$T = \frac{a}{2\pi} = \frac{\hbar a}{2\pi c k_B} \quad (3.41)$$

### 3.4 Interacting Fields in Rindler Space

Although the direct demonstration of the Unruh effect in terms of the particle density of Rindler particles in the Inertial vacuum is perfectly valid, a more general and satisfactory demonstration would be to show the equivalence between Minkowski-vacuum correlations in Rindler spacetime with a thermodynamic average over a finite temperature ensemble of Rindler particles:

$$\langle 0_M | \mathcal{T}[\phi(x_1, \dots, x_n)] | 0 \rangle = Z_T^{-1}[0] \text{Tr} \left\{ e^{-\beta H_R} \mathcal{T}[\phi(x_1, \dots, x_n)] \right\} \quad (3.42)$$

where  $x_1, \dots, x_n \in R_+$  are spacetime points in Rindler space.

This equation is the QFT equivalent of showing that the Minkowski vacuum  $|0_M\rangle$  is a mixed state with respect to Rindler particles. A general demonstration of (3.42) for interacting fields is available in various formats and levels of rigor, such as [45] and [46]. In this section we are going to briefly outline a simplified version of Unruh and Weiss's demonstration, as presented in [43].

The idea behind the demonstration of equation (3.42) is to show that both sides of the equality are analytical continuations of the same function to the same region. This would show that both functions are equal, due to the uniqueness of the analytical continuation. For the sake of clarity, let us treat the particular case of the time-ordered two-point functions  $G_2^T(\xi, \xi')$ .

For this particular demonstration, it is convenient to use a different choice of spacelike coordinate for the Rindler space  $R_+$ , given by

$$\rho = e^{a\xi} \quad \Rightarrow \quad ds^2 = \rho^2 d\tau^2 - d\rho^2 \quad (3.43)$$

We start by the trace in the right hand side: We know that in finite temperature this trace can be obtained from the Euclidean path-integral

$$Z_R^{-1}[0] \int \mathcal{D}[\phi] \phi(\tau_e, \rho_e) \phi(\tau'_e, \rho'_e) \exp \left[ - \int_0^\beta d\tau_e \int_0^\infty d\rho_e \mathcal{L}_E[\phi, \partial_\mu \phi](\tau_e, \rho_e) \right] \quad (3.44)$$

by the analytical continuation  $\tau \rightarrow -i\tau_e$ , where the Euclidean Lagrangian is written in Rindler coordinates and given by  $\mathcal{L}_E = \mathcal{L}_{(\tau=-i\tau_e)}$ . Thus we must show that the left hand side of (3.42) can also be obtained from this path-integral by an analytical continuation  $t \rightarrow -it_e$ .

In chapter 1 we have seen that the time-ordered 2-point function for the scalar field can be obtained from the analytical continuation  $t \rightarrow -it_e$  of the path-integral

$$Z^{-1}[0] \int \mathcal{D}[\phi] \phi(x_e) \phi(x'_e) \exp \left[ - \int_{-\infty}^{\infty} dt_e \int_{-\infty}^{\infty} dx_e \mathcal{L}_E[\phi, \partial_\mu \phi](t_e, x_e) \right] \quad (3.45)$$

where the Lagrangian is in Euclidean Minkowski coordinates. The demonstration of equation (3.42) is thus reduced to the problem of showing the equality of the two path-integrals above.

If we expand the exponential weight of equation (3.44)

$$\begin{aligned} - \int_0^\beta d\tau_e \int_0^\infty d\rho_e \mathcal{L}_E[\phi, \partial_\mu \phi](\tau_e, \rho_e) = \\ = - \int_0^\beta a d\tau_e \int_0^\infty \rho_e d\rho_e \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial \rho_e} \right)^2 + \frac{1}{2\rho_e^2} \left( \frac{1}{a} \frac{\partial \phi}{\partial \tau_e} \right)^2 + V(\phi) \right] \end{aligned} \quad (3.46)$$

we can see that if  $\beta = 2\pi/a$ , we can perform a polar change of variables<sup>4</sup>

$$t_e = \rho_e \sin a\tau_e \quad \text{and} \quad x_e = \rho_e \cos a\tau_e \quad (3.47)$$

and the equation for the exponential weight takes the form

$$\int_{-\infty}^{\infty} dt_e \int_{-\infty}^{\infty} dx_e \left[ \left( \frac{\partial \phi}{\partial t_e} \right)^2 + \left( \frac{\partial \phi}{\partial x_e} \right)^2 + V(\phi) \right] \quad (3.48)$$

which is equal to the Euclidean action  $S_E[\phi, \partial_\mu \phi]$  of equation (3.45). Thus the time-ordered 2-point function and the thermal average over an ensemble of Rindler particles at temperature  $T = a/2\pi$  are equivalent in the Rindler sector  $R_+$ .

Note that this demonstration is valid for all theories whose interactions can be written in the form  $V(\phi)$ . This shows that the thermal nature of the inertial vacuum state as seen by an accelerated observer is independent of interactions. Accelerated observers will see interacting theories in a finite temperature in their rest frame, and describe whatever interactions that may be possible taking into account this thermal bath of particles. In the next chapter we shall study this relationship between inertial and accelerated descriptions of a quantum system more closely.

<sup>4</sup> It is important to note here that this change of variables is **not** to be interpreted as a change in coordinates from Rindler to Minkowski ones, but only as a change in the dummy integration variables. This is a reflection of the natural  $\tau$ -periodicity ( $\tau \rightarrow \tau + 2\pi i$ ) of Rindler space  $R_+$ , which allows us to write the exponential weight of (3.44) in polar coordinates, and thus ensures the equivalence between path-integrals.

**Example 3.4.1** (Massless Thirring Model). A concrete example of equation (3.42) can be constructed for the non-trivial massless Thirring model in (1+1) spacetime dimensions. This model is solvable for the  $n$ -point functions of the field, which allows us to find the exact form of the 2-point functions in Rindler space<sup>5</sup>. The Lagrangian for the model is given by

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) \quad (3.49)$$

where  $\gamma_\mu$  are the Dirac gamma matrices in (1+1) spacetime dimensions.

The general solution for the  $n$ -point functions of this model can be found in Klaiber's work [47]. The study of the model on curved spacetime, however, was done by Birrel and Davies [48]. In their work they show that in the (1+1) dimensional Schwarzschild spacetime, the 2-point function of the field associated with the Kruskal vacuum  $|0_K\rangle$ :

$$\langle 0_K | \psi(x)\psi^*(y) | 0_K \rangle \quad (3.50)$$

is given by a thermal Green's function

$$\text{Tr}[e^{-\beta H}\psi(x)\psi^*(y)]/\text{Tr} e^{-\beta H} \quad (3.51)$$

in the region very far from the singularity  $z = 0$ , when written in terms of Schwarzschild null-coordinates. This is another demonstration of the Hawking effect about the thermal nature of black holes. To adapt this result to accelerated systems, we simply observe that the relation between Kruskal null-coordinates  $(\bar{u}, \bar{v})$  and Schwarzschild null-coordinates  $(u, v)$  in the regime far away from the singularity

$$\bar{u} = -e^{(t-z^*)/4M} \xrightarrow{z \rightarrow \infty} -e^{-u/4M} \quad \bar{v} = e^{(t+z^*)/4M} \xrightarrow{z \rightarrow \infty} e^{v/4M} \quad (3.52)$$

have the exact same form as the relation between Minkowski null-coordinates  $(\bar{U}, \bar{V})$  and Rindler null-coordinates  $(U, V)$ :

$$\bar{U} = -a^{-1}e^{-aU} \quad \bar{V} = a^{-1}e^{aV} \quad (3.53)$$

By Birrel and Davies' result, we therefore conclude that the same relationship holds for Minkowski and Rindler coordinate systems, and therefore the 2-point function of the field associated with the Minkowski vacuum  $|0_M\rangle$  is given by a thermal Green's function when written in terms of Rindler coordinates:

$$\langle 0_M | \psi(x)\psi^*(y) | 0_M \rangle = \text{Tr}[e^{-\beta H}\psi(x)\psi^*(y)]/\mathcal{Z} \quad (3.54)$$

which is the statement (3.42) about the Unruh effect.

<sup>5</sup> It is important to remark that this is only possible because we have an exact solution on the form of the correlators of the field. In other solvable models, such as the massive Thirring model which is solvable by the Bethe ansatz method, such direct approach may not be available. This is because such methods may rely on boundary conditions in order to solve the theory for the Hamiltonian eigenstates, which presents problems whenever you try to study the theory in accelerated reference frames (particularly on the consistency of the choice of boundary conditions).

## Chapter 4

# Rindler Horizon and Thermodynamics

Even though the expression (3.39) for the average number of Rindler particles in the Minkowski vacuum is the most common presentation of the Unruh effect in the literature, a slightly more rigorous one is the statement about the Minkowski vacuum expectation value of time ordered product of the fields in Rindler space:

$$\langle 0_M | \mathcal{T}(\phi(\tau_1, \xi_1) \cdots \phi(\tau_n, \xi_n)) | 0_M \rangle = \text{Tr} \left( e^{-H_R/T_U} \mathcal{T}(\phi(\tau_1, \xi_1) \cdots \phi(\tau_n, \xi_n)) \right) / \mathcal{Z} \quad (4.1)$$

The physical meaning of the equation is the following: for observers restricted to the right Rindler wedge  $R_+$ , the usual vacuum expectation value of time ordered products of fields is strictly equivalent to its thermodynamical average in Rindler space. Loosely speaking, we recover the usual statement of the Unruh effect: the Minkowski vacuum  $|0_M\rangle$  looks like a thermal state at temperature  $T_U$  for uniformly accelerated observers.

This formulation of the Unruh effect is appropriate to treat more general cases, such as interacting fields, where a solution in terms of free-field operators may not be available. In the last chapter we have shown a particular example of such formalism. The relationship (4.1) has been studied in the context of path-integrals for a large class of interacting theories [45] and for rigorous cases in axiomatic quantum field theory [46]. Although the mathematical interpretation of the thermal degrees of freedom is clear in such approaches, the most direct physical interpretation of the Rindler horizon's role in such degrees of freedom is in the perturbative analysis carried by [16] for certain classes of theories.

On this perturbative approach, the thermodynamical average in the right hand side of equation (4.1) is evaluated by finite temperature thermo field dynamical Feynman rules. As seen in section 2.4, this procedure involves an extension of the Hilbert space of the theory in order to write the trace in the thermodynamical average as an temperature-dependent vacuum expectation value in this extended space. These new degrees of freedom that

are added in the TFD formalism are identified by [16] as the degrees of freedom in the remaining regions of Minkowski space, which are invisible to the observer constrained in the right Rindler wedge region.

Such relationship between the thermal degrees of freedom as seen by the Rindler observer and the degrees of freedom from the invisible regions of Minkowski space can be further evidence that indeed the Unruh effect acts as a mechanism to maintain physical consistency between non-equivalent observers – as discussed in works such as [13, 14] – since the observers are perceiving the same degrees of freedom but are describing them differently.

At the end of the chapter we shall take this TFD approach to treat the Unruh-DeWitt detector model in such a way the roles of the invisible degrees of freedom are kept explicit. To lay ground to this future discussion and for the sake of completeness, we shall present a summarized version of [16] approach to the Unruh effect. Once again we shall consider the case of the massless scalar field in  $(1 + 1)$  spacetime dimensions. For simplicity, we shall also set the particular case of acceleration  $a = 1$  for the observer.

## 4.1 Rindler Horizon and Thermo Field Dynamics

### 4.1.1 Scalar Field Propagator in Rindler Coordinates

The first step to relating the thermal degrees of freedom in the  $R_+$  wedge with the hidden degrees of freedom outside the event horizon perceived by the accelerated observer is to divide the action of the field into four parts, corresponding to the four wedge regions of the Minkowski space – the right wedge  $R_+$ , the left wedge  $R_-$ , the future region  $F$  and the past region  $P$  (see fig. 2).

To do this we must draw coordinates for each region, which is an extension of the procedure done in section 3.2. To distinguish between the regions, we shall adopt the sub-indexes  $(\pm, F, P)$  on both coordinates and fields<sup>1</sup>:

$$\begin{aligned} R_+: \quad t &= e^{\xi_+} \sinh \tau_+ & x &= e^{\xi_+} \cosh \tau_+ \\ R_-: \quad t &= -e^{\xi_-} \sinh \tau_- & x &= -e^{\xi_-} \cosh \tau_- \\ F: \quad t &= e^{\xi_F} \cosh \tau_F & x &= e^{\xi_F} \sinh \tau_F \\ P: \quad t &= -e^{\xi_P} \cosh \tau_P & x &= -e^{\xi_P} \sinh \tau_P \end{aligned}$$

It is very important to note that all regions  $R_-$ ,  $F$  and  $P$  can be obtained by analytical continuation of Rindler coordinates in  $R_+$ :

<sup>1</sup> For this presentation we keep the coordinate convention in [16], but we remark that for the Future and Past regions, the role of timelike and spacelike coordinates change. In those regions,  $\xi$  is the timelike coordinate, and  $\tau$  the spacelike one. Under these coordinates the metric in both regions is not an static one, as it is the case of both  $R_{\pm}$ . There is a certain degree of freedom in the choice for coordinates in all these regions, but this convention, albeit confusing, helps to clear up the expressions that follows.

$$\begin{aligned}
P: \quad \tau &\rightarrow \tau - \frac{i\pi}{2} & \xi &\rightarrow \xi - \frac{i\pi}{2} \\
R_-: \quad \tau &\rightarrow \tau - i\pi & \xi &\rightarrow \xi \\
F: \quad \tau &\rightarrow \tau - \frac{3i\pi}{2} & \xi &\rightarrow \xi - \frac{i\pi}{2}
\end{aligned}$$

For a given scalar field  $\phi$  defined in Minkowski space, we can separate its action in four terms corresponding to the four regions above. Expressing the fields in each region in terms of its coordinates given above we have:

$$S = \int dt dx \mathcal{L}(\phi(t, x)) = \sum_{\pm, F, P} \int d\tau d\xi \mathcal{L}(\phi_i(\tau, \xi)) \quad (4.2)$$

where we write  $\phi_i(\tau, \xi)$  as a shorthand notation for  $\phi_i(\tau_i, \xi_i)$ .

As in the case of the previous section, we have the equality of the field representations on each respective region:

$$\phi(t, x) = \begin{cases} \phi_+(\tau, \xi), & (t, x) \in R_+ \\ \phi_-(\tau, \xi), & (t, x) \in R_- \\ \phi_F(\tau, \xi), & (t, x) \in F \\ \phi_P(\tau, \xi), & (t, x) \in P \end{cases} \quad (4.3)$$

which allows us to find the solution in each region and relate their creation and annihilation operators with the standard Minkowski ones.

The metric for each region can be calculated from the coordinate transformations:

$$g_{\pm}^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-2\xi} \quad (4.4)$$

$$g_{F/P}^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e^{-2\xi} \quad (4.5)$$

and the field equations can be written as:

$$(\partial_\tau^2 - \partial_\xi^2) \phi_i(\tau, \xi) = 0 \quad (i = \pm, F, P) \quad (4.6)$$

Following the same procedure of last section, we can write the solution for the fields  $\phi_i$  in each region terms of the particle operators associated with it:

$$\phi_i(\tau, \xi) = \int dp \left( b_i(p) g_i(p|\tau, \xi) + b_i^\dagger(p) g_i^*(p|\tau, \xi) \right) \quad (i = \pm, F, P) \quad (4.7)$$

$$g_i(p|\tau, \xi) = \frac{e^{-iE_p\tau_i + ip\xi_i}}{\sqrt{4\pi E_p}} \quad (4.8)$$

and use (4.3) to write the creation and annihilation operators  $b_i$  of the wedges in terms of their Minkowski  $a(k)$ ,  $a^\dagger(k)$  counterparts [49, 16, 44]:

$$b_\pm(p) = \frac{1}{2\pi} \int dk \theta(pk) \Gamma(-ip) E_p^{1/2} E_k^{ip-1/2} \left( e^{\pm\pi E_p/2} a(k) + e^{\mp\pi E_p/2} a^\dagger(k) \right) \quad (4.9)$$

$$\begin{aligned} b_{F/P}(p) &= \frac{1}{2\pi} \int dk \theta(pk) \Gamma(-ip) E_p^{1/2} E_k^{ip-1/2} \left( e^{\mp\pi p/2} a(k) + e^{\pm\pi p/2} a^\dagger(k) \right) \\ &= \begin{cases} b_\mp(p) & \text{if } p > 0 \\ b_\pm(p) & \text{if } p < 0 \end{cases} \end{aligned} \quad (4.10)$$

which are the extension of the Bogoliubov transformations (3.26) for the remaining regions of the Minkowski spacetime. The last equality between  $b_{F/P}$  and  $b_\pm$  is expected from considerations of causality<sup>2</sup>.

From the above expressions for the fields in  $R_\pm, F, P$ , one can proceed to calculate the Feynman propagators

$$D_{ij}(\tau, \xi; \tau', \xi') = \theta(t - t') \langle 0_M | \phi_i(\tau, \xi) \phi_j(\tau', \xi') | 0_M \rangle + \theta(t' - t) \langle 0_M | \phi_j(\tau', \xi') \phi_i(\tau, \xi) | 0_M \rangle \quad (4.11)$$

between regions  $i, j \in \{R_\pm, F, P\}$ .

For example, the propagator associated with points  $(\tau, \xi), (\tau', \xi')$  strictly contained in the right Rindler wedge  $R_+$  is given by<sup>3</sup>:

$$D_{++}(\tau, \xi; \tau', \xi') = \theta(\tau - \tau') \langle 0_M | \phi_+(\tau, \xi) \phi_+(\tau', \xi') | 0_M \rangle + (\tau, \xi \leftrightarrow \tau', \xi') \quad (4.12)$$

Using the Bogoliubov expansion of the operator  $b_+(p)$  we can find the expressions for the Minkowski vacuum expectation values of the combination of operators:

$$\langle 0_M | b_+(p) b_+(p') | 0_M \rangle = \langle 0_M | b_+^\dagger(p) b_+^\dagger(p') | 0_M \rangle = 0 \quad (4.13)$$

$$\langle 0_M | b_+^\dagger(p) b_+(p') | 0_M \rangle = n_T(p) \delta(p - p') \quad (4.14)$$

$$\langle 0_M | b_+(p) b_+^\dagger(p') | 0_M \rangle = (n_T(p) - 1) \delta(p - p') \quad (4.15)$$

and using the property of the delta function

$$\delta(x^2 - a^2) = \frac{1}{2|a|} (\delta(x - a) + \delta(x + a)) \quad (4.16)$$

<sup>2</sup> As explained in [16], "... a right (left) moving wave in the region P crosses the event horizon line  $x + t = 0$  ( $x - t = 0$ ) to enter the region  $R_+(R_-)$ . Similarly a right (left) moving wave in the region F comes from the region  $R_-(R_+)$ "

<sup>3</sup> In this region the  $t$  and  $\tau_+$  ordering coincide, since  $t = t(\tau_+)$  is a monotonically increasing function.



we can write the expression for the propagator (4.12) as:

$$D_{++}(\tau, \xi; \tau', \xi') = \int \frac{d\omega dp}{(2\pi)^2} e^{-i\omega(\tau-\tau') + ip(\xi-\xi')} \left( \frac{i}{\omega^2 - p^2 + i\epsilon} + \frac{2\pi}{e^{2\pi E_p} - 1} \delta(\omega^2 - E_p^2) \right) \quad (4.17)$$

which is composed of the usual 0-temperature free field part

$$\Delta_0(\tau, \xi; \tau', \xi') = \int \frac{d\omega dp}{(2\pi)^2} \left( \frac{i}{\omega^2 - p^2 + i\epsilon} \right) e^{-i\omega(\tau-\tau') + ip(\xi-\xi')} \quad (4.18)$$

and an on-shell temperature dependent part

$$\Delta_T(\tau, \xi; \tau', \xi') = \int \frac{d\omega dp}{(2\pi)^2} \left( \frac{2\pi}{e^{E_p/T} - 1} \delta(\omega^2 - E_p^2) \right) e^{-i\omega(\tau-\tau') + ip(\xi-\xi')} \quad (4.19)$$

with temperature  $T = 1/2\pi$  in natural units. This result can be viewed as another verification of the Unruh effect, as the inertial-vacuum expectation value of operators in Rindler space picks up a temperature dependent part  $\Delta_T$ .

Similar calculations can be done for the propagators between other regions, and the momentum-space propagators

$$D_{ij}(p) = \int d\tau d\xi D_{ij}(\tau\xi, 00) e^{i(p_0\tau + p_1\xi)} \quad (4.20)$$

take a simpler form, which can be arranged in the matrix representation [16]:

$$\begin{aligned} U(-p)D(p)U(p) = & \begin{pmatrix} \frac{i}{p^2 + i\epsilon} & 0 & 0 & 0 \\ 0 & \frac{-i}{p^2 - i\epsilon} & 0 & 0 \\ 0 & 0 & \frac{i}{p^2 + i\epsilon} & 0 \\ 0 & 0 & 0 & \frac{-i}{p^2 - i\epsilon} \end{pmatrix} + \\ & + \frac{2\pi\delta(p^2)}{e^{2\pi|p_0|} - 1} \begin{pmatrix} 1 & e^{\pi(|p_0| - p_0/2)} & e^{\pi E_p} & e^{\pi(|p_0| + p_0/2)} \\ e^{\pi(|p_0| + p_0/2)} & 1 & e^{\pi(|p_0| - p_0/2)} & e^{\pi E_p} \\ e^{\pi E_p} & e^{\pi(|p_0| + p_0/2)} & 1 & e^{\pi(|p_0| - p_0/2)} \\ e^{\pi(|p_0| - p_0/2)} & e^{\pi E_p} & e^{\pi(|p_0| + p_0/2)} & 1 \end{pmatrix} \end{aligned} \quad (4.21)$$

where  $U$  in the equation above is the diagonal matrix

$$U(p) = \text{diag}(1, e^{\pi p_1/2}, 1, e^{\pi p_1/2}) \quad (4.22)$$

#### 4.1.2 Scalar Field Propagator in Thermo Field Dynamics

To evaluate the propagator for the field in the thermo-field dynamical formalism, we must deform the path of integration to the complex plane satisfying certain conditions

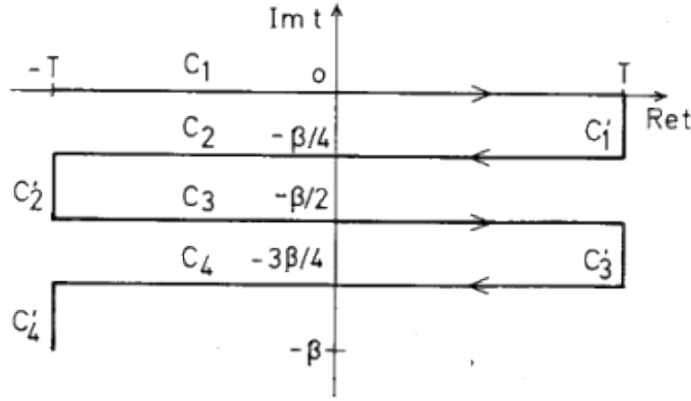


Figure 3 – Source: [16]

delineated in section 2.4. In [16] they used the path 3 in order to calculate the propagators. But we have seen that all the other regions  $P, R_-, F$  in Minkowski space can be obtained from such analytical continuation in the coordinates. It is thus better to consider a slightly different contour of integration  $\mathcal{C}$ , namely:

$$\begin{aligned}
 &(-T, \xi) \rightarrow (T, \xi) \rightarrow (T - i\beta/4, \xi) \rightarrow (T - i\beta/4, \xi - i\beta/4) \rightarrow \\
 &\rightarrow (-T - i\beta/4, \xi - i\beta/4) \rightarrow (-T - i\beta/4, \xi) \rightarrow (-T - i\beta/2, \xi) \rightarrow (T - i\beta/2, \xi) \rightarrow \\
 &\rightarrow (T - 3i\beta/4, \xi) \rightarrow (T - 3i\beta/4, \xi - i\beta/4) \rightarrow (-T - 3i\beta/4, \xi - i\beta/4) \rightarrow \\
 &\rightarrow (-T - 3i\beta/4, \xi) \rightarrow (-T - i\beta, \xi)
 \end{aligned}$$

where at the beginning and the end of every line which has opposite time ordering with respect to the real  $\tau$ -axis we also shift the  $\xi$  coordinate by segments  $\xi \rightarrow \xi \pm i\beta/4$ . This will ensure that each horizontal line represents the field in a specific region of Minkowski spacetime, namely:

$(-T, \xi) \rightarrow (T, \xi)$  : The usual field  $\phi_+$  in Rindler space  $R_+$

$(T - i\beta/4, \xi - i\beta/4) \rightarrow (-T - i\beta/4, \xi - i\beta/4)$  : The field  $\phi_-$  in the Past wedge  $P$ .

$(-T - i\beta/2, \xi) \rightarrow (T - i\beta/2, \xi)$  : The field  $\phi_-$  in the left wedge  $R_-$

$(T - 3i\beta/4, \xi - i\beta/4) \rightarrow (-T - 3i\beta/4, \xi - i\beta/4)$  : The field  $\phi_F$  in the Future wedge  $F$ .

Given this choice of contour, we can proceed to calculate the propagator  $D_\beta(\tau\xi, \tau'\xi')$  between two spacetime points in Rindler spacetime by solving

$$(\partial_\tau^2 - \partial_\xi^2)D_\beta(\tau\xi, \tau'\xi') = -\delta(\tau - \tau')\delta(\xi - \xi') \quad (4.23)$$

together with the boundary condition

$$\partial_\tau D_\beta(\tau\xi, \tau'\xi')|_{\tau=-T-i\beta} = \partial_\tau D_\beta(\tau\xi, \tau'\xi')|_{\tau=-T} \quad (4.24)$$

that ensures the periodicity of the fields. It is interesting to note here that in this case this condition is completely physical, since the analytical continuation  $\tau \rightarrow \tau - i\beta$  takes us from the right Rindler wedge into itself. Therefore equation (4.24) is nothing more than a self-consistency requirement of the argument.

Given a parametrization  $\sigma$  of the contour  $\mathcal{C}$  that increases along the path, the solution for the propagators can be obtained from the ansatz [39, 16]

$$D_\beta(\tau(\sigma)\xi(\sigma), \tau'(\sigma')\xi'(\sigma')) = D_\beta^>(\tau\xi, \tau'\xi')\theta(\sigma - \sigma') + D_\beta^<(\tau\xi, \tau'\xi')\theta(\sigma' - \sigma) \quad (4.25)$$

where the equations for the propagator gives us equations for  $D_\beta^\lessgtr(\tau\xi, \tau'\xi')$ :

$$(\partial_t^2 - \partial_\xi^2)D_\beta^\lessgtr(\tau\xi, \tau'\xi') = 0 \quad (4.26)$$

$$\partial_\tau D_\beta^>(\tau\xi, \tau'\xi') - \partial_\tau D_\beta^<(\tau\xi, \tau'\xi') = -\delta(\xi - \xi') \quad (4.27)$$

$$\partial_\tau D_\beta^\lessgtr((-T - i\beta)\xi, \tau'\xi') = \partial_\tau D_\beta^\lessgtr(-T\xi, \tau'\xi') \quad (4.28)$$

Fourier transforming with respect to  $\xi - \xi'$  we obtain the solution for the propagator  $D_\beta(\tau\xi, \tau'\xi')$  as

$$\begin{aligned} D_\beta(\tau(\sigma)\xi(\sigma), \tau'(\sigma')\xi'(\sigma')) &= -\frac{i}{2\pi} \int \frac{dp}{2|p|} \frac{1}{1 - e^{-\beta|p|}} \\ &\times e^{ip(\xi - \xi')} \left[ \left( e^{-\beta|p|} e^{ip|(\tau - \tau')|} + e^{-ip|(\tau - \tau')|} \right) \theta(\sigma - \sigma') + \right. \\ &\left. + \left( e^{ip|(\tau - \tau')|} + e^{-\beta|p|} e^{-ip|(\tau - \tau')|} \right) \theta(\sigma' - \sigma) \right] \end{aligned} \quad (4.29)$$

With this expression we can calculate the propagator  $D_\beta$  for any patch of the contour  $\mathcal{C}$ . Remembering that when asymptotically in T the contributions of the vertical paths  $(\tau, \xi) \rightarrow (\tau + iX, \xi)$  and  $(\tau, \xi) \rightarrow (\tau, \xi + iX)$  that are used to shift coordinates are negligible, the contour  $\mathcal{C}$  contributes to four thermo-field dynamical fields corresponding to its horizontal paths:  $\phi_1, \phi_2, \phi_3$  and  $\phi_4$ . The field corresponding to physical degrees of freedom is the  $\phi_1$  field, which corresponds to the real-time path in the contour  $\mathcal{C}$ .

For example, let us calculate the propagator  $D_\beta[2, 2](\tau\xi, 00)$  corresponding to field  $\phi_2$ : Let us assume that  $\tau > 0$ . In this patch of the contour,  $\tau$ -ordering is the oposite of  $\sigma$ -ordering, and therefore if  $\tau > 0$  only the  $\theta(-\sigma)$  contributes to the propagator. Remembering that in this contour patch  $(\tau(\sigma), \xi(\sigma)) = (\tau - i\beta/4, \xi - i\beta/4)$ , the propagator for  $\tau > 0$  can be written as

$$\theta(\tau)D_\beta[2, 2](\tau\xi) = -\frac{i}{2\pi} \int \frac{dp}{2E_p} \frac{e^{\beta p/4}}{e^{\beta E_p} - 1} e^{ip\xi} \left( e^{\beta E_p} e^{iE_p(\tau - i\beta/4)} + e^{-iE_p(\tau - i\beta/4)} \right) \quad (4.30)$$

where we abuse the notation  $E_p = |p|$ .

The analogous calculation can be done for the  $\tau < 0$  case:

$$\theta(-\tau)D_\beta[2, 2](\tau\xi) = -\frac{i}{2\pi} \int \frac{dp}{2E_p} \frac{e^{\beta p/4}}{e^{\beta E_p} - 1} e^{ip\xi} \left( e^{iE_p(\tau-i\beta/4)} + e^{\beta E_p} e^{-iE_p(\tau-i\beta/4)} \right) \quad (4.31)$$

which completely specifies the propagator

$$D_\beta[2, 2](\tau\xi) = \theta_{C_2}(\tau)D_\beta[2, 2](\tau\xi) + \theta_{C_2}(-\tau)D_\beta[2, 2](\tau\xi) \quad (4.32)$$

on the horizontal complex-contour patch  $\mathcal{C}_2 = \{(-T - i\beta/4, +T - i\beta/4), \xi \in \mathbb{R} - i\beta/4\}$ .

Writing the full propagator (4.29) for each horizontal patch of contour  $\mathcal{C}$  in momentum space

$$D_\beta[ij](p) = \int d\tau d\xi D_\beta[ij](\tau\xi, 00) e^{i(p_0\tau + p\xi)} \quad (4.33)$$

we find that for  $\beta = 2\pi$  the thermo-field propagator  $D_\beta[ij](p)$  and the Rindler-space propagator  $D[ij](p)$  of equation (4.21) coincide:

$$D[ij](p) = D_\beta[ij](p) \quad (4.34)$$

and now we can precisely identify the thermo-fields  $\phi_1, \phi_2, \phi_3, \phi_4$  with the corresponding fields in the right, past, left and future wedges respectively.

This result is slightly different from [16], since we have modified the TFD integration contour in order for the horizontal patches to correspond with the explicit analytical continuations of the Rindler coordinates to the other regions of Minkowski spacetime. This correspondence is associated with the shift  $\xi \rightarrow \xi - i\beta/4$  in the TFD contour for the patches with opposite  $\tau$ -ordering, which by the properties of the Fourier transform correspond to the effect of multiplying the propagator by a factor of  $\exp(\beta p/4)$ , which is the extra factors  $U(p)$  that are absent in Horibe et. al's TFD propagator.

In this contour, the correspondence between TFD and QFT in accelerated reference frames is complete, and it carries on to any possible perturbative analysis based on the massless scalar field in (1+1) dimensions.

This correspondence further supports the view that the Unruh effect is a “coriolis”-type phenomena which appears in order to maintain consistency between physics as seen by an inertial and a comoving observer. Each observer is describing the same degrees of freedom, namely the field degrees of freedom at each point  $p \in \mathcal{M}_{ink}$  of Minkowski space, but are assigning different interpretations to them. The inertial observer sees them as normal degrees of freedom that contributes to its path-integral representation of the system, while the accelerated observer describes them as invisible degrees of freedom that are necessary in order to describe the finite-temperature effects that he sees in his restframe.

The relationship between processes from the point of view of inertial and accelerated observers can be directly obtained from the propagator equality (4.34). All perturbative diagrams constructed by the inertial observer in order to calculate an observable can be directly translated to the diagrams the accelerated observer sees in its rest frame. The only difference is that the accelerated observer perceives only  $\phi_+ \equiv \phi(x \in R_+)$  as physical degrees of freedom, and all the others as fictitious fields that appear in his description of the system.

## 4.2 Example: Internal Particle Detectors

The thermal nature of the Minkowski vacuum with respect to accelerated observers becomes especially important when dealing with the physics of semiclassical detectors accelerating through fields. In such cases, inertial and accelerated observers will describe the same phenomena in distinct but entirely equivalent ways – the choice of reference frame being only a matter of convenience. For particular class of systems we may use the direct association of the thermal degrees of freedom perceived by the accelerated observer with the hidden degrees of freedom beyond the Rindler horizon, perceived by the inertial observer, to specify the relationship between processes in both reference frames.

We define as an internal particle detector the pair  $(Q, x_Q)$ , where  $Q$  is a localized degree of freedom (usually taken to be a simple two-level system, in the case of Unruh-DeWitt detectors) and  $x_Q$  a particular worldline on spacetime. In this section we shall work only with a particular class of worldlines, namely those entirely contained in the right Rindler wedge  $R_+$ , which we name internal Rindler detectors. This ensures that both inertial and accelerated observers have access to the entire history of the particle, and avoid the cases where the particle crosses the Rindler horizon.

### 4.2.1 Detector Model

We are going to work with the more general form of internal Rindler detectors as presented in [50]. There the particle-field system is described by the action

$$S = S_0(\phi) + S_I(\phi, J_Q) \quad (4.35)$$

$$S_I(\phi, J_Q) = \int dx \sqrt{|g(x)|} J_Q(x) \phi(x) \quad (4.36)$$

with  $S_0$  being the action for a free scalar field  $\phi$  and  $S_I$  the coupling between the field and the detector. In this context  $J_Q$  represents the current for a particular transition of the degree of freedom  $Q(E_Q \rightarrow E'_Q)$ :

$$J_Q(x) = c(\tau) \langle E'_Q | Q(0) | E_Q \rangle e^{i\Delta E \tau} \delta(\mathbf{x} - \mathbf{x}_Q) / \sqrt{|g(x)|} \quad (4.37)$$

with  $\tau$  the detector's proper time,  $c$  a smooth function that mediates the interaction coupling strength and  $g$  the spacetime metric.

It is convenient to work with the current of a particular process  $Q(E_Q \rightarrow E'_Q)$  because we are interested in the field's reaction, not in the internal processes of the detector itself. An inertial observer describes the amplitude of a particle emission process in first-order perturbation on the detector-field coupling as

$$\mathcal{A}_{Em}^M(k) = \langle k | S_I(\phi, J_Q) | 0_M \rangle = \langle 0_M | a(k) S_I | 0_M \rangle \quad (k > 0) \quad (4.38)$$

From the Minkowski solution for the free scalar field  $\phi$  we can write an expression for the amplitude solely in terms of the current  $J_Q$  of the detector:

$$\phi(x) = \int \frac{dk}{\sqrt{4\pi E_k}} (a(k)e^{-ikx} + a^\dagger(k)e^{ikx}) \quad (4.39)$$

$$\mathcal{A}_{Em}^M(k) = \frac{1}{\sqrt{4\pi E_k}} \int dx J_Q(x) e^{ikx} \quad (4.40)$$

With this expression we can calculate, given the associated detector current for a particular internal process  $J_Q$ , the probability of a particle emission with momentum  $k$  with respect to an inertial observer.

## 4.2.2 Thermo Field Dynamical Approach

To analyze this process from the point of view of accelerated observers, we shall take a different approach than [50]. By applying the relationship between the Unruh effect and thermo field dynamics, we ensure that the role of the hidden degrees of freedom beyond the Rindler horizon remain explicit throughout the derivation.

For consistency of notation, we first must rewrite the Rindler solution's creation and annihilation operators from both the right(+) and left(-) Rindler wedge

$$\phi_\pm(\tau, \xi) = \int dp (b_\pm(p) g_\pm(\tau, \xi) + b_\pm^\dagger(p) g_\pm^*(\tau, \xi)) \quad (4.41)$$

in a form consistent with TFD. The subindex refers to the region on which field is referring – the same way as in the previous section. To do this we use the solutions (4.9) for the Bogoliubov coefficients in  $R_\pm$  to derive relations between  $b_+$  and  $b_-$ :

$$\left[ b_+(p) - e^{-E_p \pi/a} b_-^\dagger(p) \right] |0_M\rangle = 0 \quad (4.42)$$

$$\left[ b_-(p) - e^{-E_p \pi/a} b_+^\dagger(p) \right] |0_M\rangle = 0 \quad (4.43)$$

rewriting  $\exp(-E_p \pi/a)$  in terms of the (bosonic) thermal factor

$$\gamma_B(E_p) = \frac{1}{e^{\beta E_p} - 1} \quad \beta = \frac{2\pi}{a} \quad (4.44)$$

we have

$$e^{-E_p \pi/a} = \left( \frac{\gamma_B(E_p)}{1 + \gamma_B(E_p)} \right)^{1/2} \quad (4.45)$$

Equations (4.42) and (4.43) can then be rewritten in the familiar thermo field dynamical form

$$b_\beta(p) |0_M\rangle = 0, \quad b_\beta(p) = \sqrt{1 + \gamma_B(E_p)} b_+(p) - \sqrt{\gamma_B(E_p)} b_-^\dagger(p) \quad (4.46)$$

$$\tilde{b}_\beta(p) |0_M\rangle = 0, \quad \tilde{b}_\beta(p) = \sqrt{1 + \gamma_B(E_p)} b_-(p) - \sqrt{\gamma_B(E_p)} b_+^\dagger(p) \quad (4.47)$$

From the above equations we can identify the Minkowski vacuum  $|0_M\rangle$ , viewed by the inertial observer, as the Rindler thermofield vacuum  $|0(\beta)\rangle$ , as perceived by the accelerating one.

Since we are assuming that the detector's worldline satisfy  $J_Q(x) \subset R_+$  the detector is only sensitive to  $\phi_+$  excitations, and the  $\phi_-$  sector is completely invisible as far as the detector is concerned. Thus as in standard TFD  $b_\beta(p)$  is the operator for physical excitations, while  $\tilde{b}_\beta(p)$  is the operator for the fictitious holes in TFD.

The Minkowski amplitude (4.38) can now be rewritten in terms of the Rindler thermofield operators  $b_\beta(p)$  and  $\tilde{b}_\beta(p)$ . We can express the Minkowski annihilation operator  $a(k)$  in terms of its TFD decomposition

$$a(k) = \int_0^\infty \frac{dp}{\sqrt{1 + \gamma_B(p)}} \left( \alpha_{pk}^R b_\beta(p) + \alpha_{pk}^L \tilde{b}_\beta(p) \right) \quad (k > 0) \quad (4.48)$$

with  $\alpha_{pk}^{R,L}$  the Bogoliubov coefficients relating the Minkowski and the (right, left) Rindler wedges. The expression for the Minkowski emission amplitude is then:

$$\begin{aligned} \mathcal{A}_{Em}^M(k) &= \langle 0_M | a(k) S_I | 0_M \rangle = \\ &= \langle 0(\beta) | \int_0^\infty \frac{dp}{\sqrt{1 + \gamma_B(E_p)}} \left( \alpha_{pk}^R b_\beta(p) + \alpha_{pk}^L \tilde{b}_\beta(p) \right) S_I(\phi, J_Q) | 0(\beta) \rangle \end{aligned}$$

Defining the emission and absorption amplitudes with respect to Rindler thermofield particles

$$\mathcal{A}_{Em}^\beta(p) = \langle 0(\beta) | b_\beta(p) S_I | 0(\beta) \rangle \quad (4.49)$$

$$\mathcal{A}_{Ab}^\beta(p) = \langle 0(\beta) | \tilde{b}_\beta(p) S_I | 0(\beta) \rangle \quad (4.50)$$

we have the relationship between the amplitudes associated with the two observers:

$$\mathcal{A}_{Em}^M(k) = \int_0^\infty \frac{dp}{\sqrt{1 + \gamma_B(p)}} \left( \alpha_{pk}^R \mathcal{A}_{Em}^\beta + \alpha_{pk}^L \mathcal{A}_{Ab}^\beta(p) \right) \quad (4.51)$$

The Minkowski emission probability  $\mathcal{P}_{Em}^M = \int dk |\mathcal{A}_{Em}^M(k)|^2$  is related to the Rindler probabilities

$$\mathcal{P}_{Em}^M = \int_0^\infty dp \left( |\mathcal{A}_{Em}^\beta|^2 + |\mathcal{A}_{Ab}^\beta|^2 \right) = \mathcal{P}_{Em}^\beta + \mathcal{P}_{Ab}^\beta \quad (4.52)$$

which is the simple sum of the corresponding TFD emission and absorption probabilities. Here the absorption part is the explicit creation of a hole in the thermofield vacuum  $|0(\beta)\rangle$ , while the emission part is the creation of a physical excitation on it. This is consistent with the standard QFT calculation presented in [50], as the TFD amplitudes  $\mathcal{A}_{Em/Ab}^\beta(p)$  can be written in terms of the Rindler amplitudes  $\mathcal{A}_{Em/Ab}^R(p)$  as

$$\mathcal{A}_{Em}^\beta(p) = \sqrt{1 + \gamma_B(E_p)} \mathcal{A}_{Em}^R(p) \quad \mathcal{A}_{Em}^R(p) = \langle 0_R | b_+(p) S_I | 0_R \rangle \quad (4.53)$$

$$\mathcal{A}_{Ab}^\beta(p) = \sqrt{\gamma_B(E_p)} \mathcal{A}_{Ab}^R(p) \quad \mathcal{A}_{Ab}^R(p) = \langle 0_R | S_I b_+^\dagger(p) | 0_R \rangle \quad (4.54)$$

The explicit relationship between the Rindler processes as seen by the observer confined to the right Rindler wedge  $R_+$  and their TFD interpretation can be further seen by making use of the properties

$$b_\beta^\dagger(p) |0(\beta)\rangle = \frac{1}{\sqrt{\gamma_B(E_p)}} b_-(p) |0(\beta)\rangle = \frac{1}{\sqrt{1 + \gamma_B(E_p)}} b_+^\dagger(p) |0(\beta)\rangle \quad (4.55)$$

$$\tilde{b}_\beta^\dagger(p) |0(\beta)\rangle = \frac{1}{\sqrt{\gamma_B(E_p)}} b_-^\dagger(p) |0(\beta)\rangle = \frac{1}{\sqrt{1 + \gamma_B(E_p)}} b_+(p) |0(\beta)\rangle \quad (4.56)$$

which shows that in fact the creation of a Rindler particle on Minkowski vacuum corresponds to the annihilation of a left wedge Rindler particle, indeed acting as the fictitious holes in TFD. Such relationship between trans-horizon degrees of freedom is a direct consequence of relations 4.42 and 4.43.



## Chapter 5

# Conclusions and Discussions

In this thesis we have studied quantum fields in accelerated reference frames. We have discussed the thermal nature of the inertial vacuum from the point of view of an accelerated observer, and saw how the foundations of quantum field theory itself dictates how inequivalence between observers leads to the ambiguous concept of particle excitations. This serves to show how our intuition breaks down when we start treating more general possibilities for the spacetime manifold of the universe. The conceptual difficulties present in this case are only a taste of how weird things can be if we let go of Poincaré symmetry or the existence of a global timelike Killing vector. Such concepts are what supports our intuition of the world, and in their absence we must rely on concrete mathematical constructions in order to truly understand physics.

We have also shown the direct connection between the thermal degrees of freedom perceived by the accelerated observer with the invisible degrees of freedom that are accessible to the inertial frame. This connection is only a particular case of the more general connection between spacetimes with horizons and thermodynamics, which still is an active topic of research to this day. The appearance of thermal degrees of freedom in the accelerated observers rest frame is a direct consequence of the tracing over the degrees of freedom which he cannot see. The mechanism of such effect is exactly analogous to the more famous Hawking effect for black holes<sup>1</sup>, where an asymptotic observer does not have access to the degrees of freedom inside the hole. In fact, since the spacetime of a Schwarzschild black hole is locally Rindler near the event horizon, all of the discussions in the thesis also apply for near-black-hole systems.

This connection between thermal and trans-horizon degrees of freedom is concretely implemented in the connection between TFD propagators in Rindler space and the Feyn-

---

<sup>1</sup> This is expected, since by the equivalence principle gravity and acceleration are two sides of the same coin.

man propagators between trans-horizon regions. This relation allowed us to solve the problem of the accelerated semi-classical detector purely by means of thermo-field dynamical methods. This helped us better understand the role of the Unruh effect in the dynamics of semi-classical systems as a phenomenon that is necessary to maintain physics consistent across inequivalent observers. Even though it is reasonable to expect different levels of difficulty between both approaches when applied to real systems, it is clear however that they both are entirely equivalent approaches to the same problem.

It is worth noting that this Unruh-TFD formalism is only applicable to uniformly accelerated observers. Even though the semiclassical field-detector system under study might be allowed to fluctuate due to particle interactions, this Unruh-TFD procedure is only able to connect the physical descriptions of the system between inertial and uniformly accelerating points of view. A generalization of such procedure might be possible by applying a non-equilibrium formulation of thermo-field dynamics, such as [51, 52, 53, 54], and is therefore another possibility for future studies.

Further implications of this view in other areas is possibly one of the most interesting and promising topic for our future investigations. From early thermalization of hadrons to electromagnetic radiation emitted by accelerated electrons, we hope that the study of accelerated systems can help us better understand physics in the frontier of quantum and semi-classical scales, as well as the frontier of our knowledge about the universe.

# Bibliography

- [1] Peter Brockway Arnold. Quark-gluon plasmas and thermalization. *Int. J. Mod. Phys.*, E16:2555–2594, 2007. Cited on page 10.
- [2] Ulrich W. Heinz and Peter F. Kolb. Two RHIC puzzles: Early thermalization and the HBT problem. In *18th Winter Workshop on Nuclear Dynamics (WWND 2002) Nassau, Bahamas, January 20-22, 2002*, 2002. Cited on page 10.
- [3] F. Becattini, M. Gazdzicki, and J. Sollfrank. On chemical equilibrium in nuclear collisions. *Eur. Phys. J.*, C5:143–153, 1998. Cited on page 10.
- [4] Francesco Becattini. An introduction to the statistical hadronization model. *arXiv preprint arXiv:0901.3643*, 2009. Cited on page 10.
- [5] F Becattini. Universality of thermal hadron production in pp, pp and e e- collisions. *arXiv preprint hep-ph/9701275*, 1996. Cited on page 10.
- [6] Dmitri Kharzeev and Kirill Tuchin. From color glass condensate to quark–gluon plasma through the event horizon. *Nuclear Physics A*, 753(3-4):316–334, 2005. Cited on page 10.
- [7] Paolo Castorina, D Kharzeev, and Helmut Satz. Thermal hadronization and hawking–unruh radiation in qcd. *The European Physical Journal C-Particles and Fields*, 52(1):187–201, 2007. Cited on page 10.
- [8] F Becattini, P Castorina, J Manninen, and Helmut Satz. The thermal production of strange and non-strange hadrons in e+ e- collisions. *The European Physical Journal C*, 56(4):493–510, 2008. Cited on page 10.
- [9] Paolo Castorina, Alfredo Iorio, and Helmut Satz. Hadron freeze-out and unruh radiation. *International Journal of Modern Physics E*, 24(07):1550056, 2015. Cited on page 10.
- [10] Stephen A Fulling. Nonuniqueness of canonical field quantization in riemannian space-time. *Physical Review D*, 7(10):2850, 1973. Cited on page 10.

- [11] William G Unruh. Notes on black-hole evaporation. *Physical Review D*, 14(4):870, 1976. Cited on page 10.
- [12] A Higuchi, GEA Matsas, and D Sudarsky. Bremsstrahlung and zero-energy rindler photons. *Physical Review D*, 45(10):R3308, 1992. Cited on page 11.
- [13] A Higuchi, GEA Matsas, and D Sudarsky. Bremsstrahlung and fulling-davies-unruh thermal bath. *Physical Review D*, 46(8):3450, 1992. Cited 2 times on pages 11 and 46.
- [14] Daniel AT Vanzella and George EA Matsas. Decay of accelerated protons and the existence of the fulling-davies-unruh effect. *Physical review letters*, 87(15):151301, 2001. Cited 2 times on pages 11 and 46.
- [15] Pisin Chen and Toshi Tajima. Testing unruh radiation with ultraintense lasers. *Physical Review Letters*, 83(2):256, 1999. Cited on page 11.
- [16] Minoru Horibe, Akio Hosoya, and Noboru Yamamoto. On perturbation expansion of field theory in accelerated system and the thermofield dynamics. *Progress of theoretical physics*, 74(6):1299–1315, 1985. Cited 10 times on pages 11, 31, 36, 45, 46, 48, 49, 50, 51, and 52.
- [17] Norma Sánchez. Analytic mappings: A new approach to quantum field theory in accelerated frames. *Physical Review D*, 24(8):2100, 1981. Cited on page 11.
- [18] Dennis W Sciama, P Candelas, and D Deutsch. Quantum field theory, horizons and thermodynamics. *Advances in Physics*, 30(3):327–366, 1981. Cited on page 11.
- [19] Werner Israel. Thermo-field dynamics of black holes. *Physics Letters A*, 57(2):107–110, 1976. Cited on page 11.
- [20] Gary W Gibbons and Stephen W Hawking. Cosmological event horizons, thermodynamics, and particle creation. *Physical Review D*, 15(10):2738, 1977. Cited on page 11.
- [21] Carlos E Laciana. Quantum field theory in curved space-time as thermo field dynamics. *General relativity and gravitation*, 26(4):363–378, 1994. Cited on page 11.
- [22] George EA Matsas and Daniel AT Vanzella. The fulling–davies–unruh effect is mandatory: the proton’s testimony. *International Journal of Modern Physics D*, 11(10):1573–1577, 2002. Cited on page 11.
- [23] Steven Weinberg. *The quantum theory of fields*, volume 1. Cambridge university press, 1995. Cited 3 times on pages 15, 19, and 22.
- [24] Elcio Abdalla and M Cristina B Abdalla. *Non-perturbative methods in 2 dimensional quantum field theory*. World Scientific, 2001. Cited 2 times on pages 17 and 37.

- [25] Huzihiro Araki. *Mathematical theory of quantum fields*, volume 101. Oxford University Press on Demand, 1999. Cited on page 17.
- [26] Michael Creutz. *Quarks, gluons and lattices*, volume 8. Cambridge University Press, 1983. Cited on page 17.
- [27] István Montvay and Gernot Münster. *Quantum fields on a lattice*. Cambridge University Press, 1997. Cited on page 17.
- [28] Michael E Peskin, Daniel V Schroeder, and Emil Martinec. An introduction to quantum field theory, 1996. Cited 2 times on pages 19 and 22.
- [29] Stephen A Fulling. *Aspects of quantum field theory in curved spacetime*, volume 17. Cambridge university press, 1989. Cited on page 20.
- [30] Steven Weinberg. The cosmological constant problem. *Reviews of modern physics*, 61(1):1, 1989. Cited on page 20.
- [31] Raymond F Streater and Arthur S Wightman. *PCT, spin and statistics, and all that*. Princeton University Press, 2016. Cited on page 22.
- [32] Richard Phillips Feynman. Space-time approach to non-relativistic quantum mechanics. *Reviews of Modern Physics*, 20(2):367, 1948. Cited on page 22.
- [33] Walter Greiner and Joachim Reinhardt. *Field quantization*. Springer Science & Business Media, 2013. Cited on page 22.
- [34] James Glimm and Arthur Jaffe. *Quantum physics: a functional integral point of view*. Springer Science & Business Media, 2012. Cited on page 22.
- [35] Gert Roepstorff. *Path integral approach to quantum physics: an introduction*. Springer Science & Business Media, 2012. Cited on page 22.
- [36] Joseph I Kapusta and Charles Gale. *Finite-temperature field theory: Principles and applications*. Cambridge University Press, 2006. Cited 2 times on pages 27 and 28.
- [37] Hiroomi Umezawa, Hiroshi Matsumoto, and Masashi Tachiki. *Thermo field dynamics and condensed states*. North-Holland Publishing Company, 1982. Cited on page 27.
- [38] Yasushi Takahashi and Hiroomi Umezawa. Thermo field dynamics. *International journal of modern Physics B*, 10(13n14):1755–1805, 1996. Cited on page 28.
- [39] Antti J Niemi and Gordon W Semenoff. Finite-temperature quantum field theory in minkowski space. *Annals of Physics*, 152(1):105–129, 1984. Cited 2 times on pages 30 and 51.

- [40] Antti J Niemi and Gordon W Semenoff. Thermodynamic calculations in relativistic finite-temperature quantum field theories. *Nuclear Physics B*, 230(2):181–221, 1984. Cited on page 30.
- [41] LH Ford. Quantum field theory in curved spacetime. *arXiv preprint gr-qc/9707062*, 1997. Cited on page 32.
- [42] LD Landau and EM Lifshits. Theoretical physics, vol. 1: Mechanics. *English translation, Pergamon Press, Book p*, 9099, 1988. Cited on page 35.
- [43] Luis CB Crispino, Atsushi Higuchi, and George EA Matsas. The unruh effect and its applications. *Reviews of Modern Physics*, 80(3):787, 2008. Cited 2 times on pages 40 and 42.
- [44] VF Mukhanov and S Winitzki. Quantum fields in classical backgrounds, 2004. Cited 2 times on pages 40 and 48.
- [45] William G Unruh and Nathan Weiss. Acceleration radiation in interacting field theories. *Physical Review D*, 29(8):1656, 1984. Cited 2 times on pages 42 and 45.
- [46] Geoffrey L Sewell. Quantum fields on manifolds: Pct and gravitationally induced thermal states. *Annals of Physics*, 141(2):201–224, 1982. Cited 2 times on pages 42 and 45.
- [47] B. Klaiber. Lectures in theoretical physics. 1968. Cited on page 44.
- [48] ND Birrell and PCW Davies. Massless thirring model in curved space: thermal states and conformal anomaly. *Physical Review D*, 18(12):4408, 1978. Cited on page 44.
- [49] Michael Soffel, Berndt Müller, and Walter Greiner. Dirac particles in rindler space. *Physical Review D*, 22(8):1935, 1980. Cited on page 48.
- [50] George EA Matsas. Rindler and minkowski particles relationship revisited. *Physics Letters B*, 380(1):24–28, 1996. Cited 3 times on pages 53, 54, and 56.
- [51] Toshihiko Arimitsu and Hiroomi Umezawa. A general formulation of nonequilibrium thermo field dynamics. *Progress of theoretical physics*, 74(2):429–432, 1985. Cited on page 58.
- [52] Toshihico Arimitsu and Hiroomi Umezawa. Non-equilibrium thermo field dynamics. *Progress of theoretical physics*, 77(1):32–52, 1987. Cited on page 58.
- [53] Toshihico Arimitsu. A canonical formalism of dissipative quantum systems. non-equilibrium thermo field dynamics. *Condensed Matter Physics*, 4:26–88, 1994. Cited on page 58.

- 
- [54] Dmitrii Nikolaevich Zubarev and Mikhail Vasil'evich Tokarchuk. Nonequilibrium thermofield dynamics and the nonequilibrium statistical operator method. *Theoretical and Mathematical Physics*, 88(2):876–893, 1991. Cited on page 58.